LIPSCHITZ AND UNIFORMLY CONTINUOUS REDUCIBILITIES ON ULTRAMETRIC POLISH SPACES

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ABSTRACT. We analyze the reducibilities induced by, respectively, uniformly continuous, Lipschitz, and nonexpansive functions on arbitrary ultrametric Polish spaces, and determine whether under suitable set-theoretical assumptions the induced degree-structures are well-behaved.

1. INTRODUCTION

Throughout the paper, we work in the usual Zermelo-Frænkel set theory ZF, plus the Axiom of Dependent Choices over the reals $\mathsf{DC}(\mathbb{R})$. Let X be a Polish space, and let $\mathcal{F}$ be a *reducibility (on X)*, that is a collection of functions from X to itself closed under composition and containing the identity $\mathrm{id} = \mathrm{id}_X$. Given $A, B \subseteq X$, we say that A is *reducible* to B if and only if

$$A = f^{-1}(B) \text{ for some } f \colon X \to X,$$

and that A is *$\mathcal{F}$-reducible* to B ($A \leq_{\mathcal{F}} B$ in symbols) if A is reducible to B *via a function in $\mathcal{F}$*. Notice that clearly $A \leq_{\mathcal{F}} B \iff \neg A \leq_{\mathcal{F}} \neg B$ (where, to simplify the notation, we set $\neg A = X \setminus A$ whenever the underlying space X is clear from the context). Since $\mathcal{F}$ is a reducibility on X, the relation $\leq_{\mathcal{F}}$ is a preorder which can be used to measure the "complexity" of subsets of X: in fact, if $\mathcal{F}$ consists of reasonably simple functions, the assertion "$A \leq_{\mathcal{F}} B$" may be understood as "the set A is not more complicated than the set B" — to test whether a given $x \in X$ belongs to A or not, it is enough to pick a witness $f \in \mathcal{F}$ of $A \leq_{\mathcal{F}} B$, and then check whether $f(x) \in B$ or not. This suggests that the reducibility $\mathcal{F}$ may be used to form a hierarchy of subsets of X in the following way. Say that $A, B \subseteq X$ are *$\mathcal{F}$-equivalent* ($A \equiv_{\mathcal{F}} B$ in symbols) if $A \leq_{\mathcal{F}} B \leq_{\mathcal{F}} A$. Since $\equiv_{\mathcal{F}}$ is the equivalence relation canonically induced by $\leq_{\mathcal{F}}$, we can consider the *$\mathcal{F}$-degree* $[A]_{\mathcal{F}} = \{B \subseteq X \mid A \equiv_{\mathcal{F}} B\}$ of a given $A \subseteq X$, and then order the collection $\mathsf{Deg}(\mathcal{F}) = \{[A]_{\mathcal{F}} \mid A \subseteq X\}$ of such $\mathcal{F}$-degrees using the quotient of $\leq_{\mathcal{F}}$, namely setting $[A]_{\mathcal{F}} \leq [B]_{\mathcal{F}} \iff A \leq_{\mathcal{F}} B$ for every $A, B \subseteq X$. The resulting structure $\mathsf{Deg}(\mathcal{F}) = (\mathsf{Deg}(\mathcal{F}), \leq)$ is then called *$\mathcal{F}$-hierarchy on X*. When considering the restriction $\mathsf{Deg}_{\boldsymbol{\Gamma}}(\mathcal{F})$ of such structure to the $\mathcal{F}$-degrees of sets in a given $\boldsymbol{\Gamma} \subseteq \mathscr{P}(X)$, we speak of *$\mathcal{F}$-hierarchy on $\boldsymbol{\Gamma}$-subsets of X*.

In his Ph.D. thesis [Wad83], Wadge considered the case when X is the Baire space $^\omega\omega$ (i.e. the space of all ω-sequences of natural numbers endowed with the product of the discrete topology on ω) and $\mathcal{F}$ is either the set $\mathsf{W} = \mathsf{W}(X)$ of all continuous functions, or the set $\mathsf{L}(\bar{d})$ of all functions which are nonexpansive with respect to the usual metric $\bar{d}$ on $^\omega\omega$ (see Section 2 for the definition). Using game-theoretical methods, he was able to show that in both cases the $\mathcal{F}$-hierarchy on Borel subsets of $X = {}^\omega\omega$ is *semi-well-ordered*, that is:

(1) it is *semi-linearly ordered*, i.e. either $A \leq_\mathcal{F} B$ or $\neg B \leq_\mathcal{F} A$ for all Borel $A, B \subseteq X$;

(2) it is *well-founded*.

Notice that the *Semi-Linear Ordering principle for $\mathcal{F}$* (briefly: $\mathsf{SLO}^\mathcal{F}$) defined in (1) implies that antichains have size at most 2, and that they are of the form $\{[A]_\mathcal{F}, [\neg A]_\mathcal{F}\}$ for some $A \subseteq X$ such that $A \nleq_\mathcal{F} \neg A$ (sets with this last property are called $\mathcal{F}$-*nonselfdual*, while the other ones are called $\mathcal{F}$-*selfdual*: since $\mathcal{F}$-selfduality is $\equiv_\mathcal{F}$-invariant, a similar terminology will be applied to the $\mathcal{F}$-degree of A as well). This in particular means that if we further identify each $\mathcal{F}$-degree $[A]_\mathcal{F}$ with its *dual* $[\neg A]_\mathcal{F}$ we get a linear ordering, which is also well-founded when (2) holds.

A semi-well-ordered hierarchy is practically optimal as a measure of complexity for (Borel subsets of) X: by well-foundness, we can associate to each $A \subseteq X$ an ordinal rank (the $\mathcal{F}$-*rank of A*), and antichains are of minimal size.[1] In fact, in [MRSS12, MR12] it is proposed to classify arbitrary $\mathcal{F}$-hierarchies on corresponding topological spaces X according to whether they provide a good measure of complexity for subsets of X. This led to the following definition.

Definition 1.1. Let $\mathcal{F}$ be a reducibility on a (topological) space X, and let $\mathbf{\Gamma} \subseteq \mathscr{P}(X)$. The $\mathcal{F}$-hierarchy $\mathsf{Deg}_\mathbf{\Gamma}(\mathcal{F})$ on $\mathbf{\Gamma}$-subsets of X is called:

- *very good* if it is semi-well-ordered;
- *good* if it is a well-quasi-order, i.e. all its antichains and descending chains are finite;
- *bad* if it contains infinite antichains;
- *very bad* if it contains both infinite antichains and infinite descending chains.

Since the pioneering work of Wadge, many other $\mathcal{F}$-hierarchies on the Baire space $^\omega\omega$ (or, more generally, on *zero-dimensional* Polish space) have been considered in the literature [VW78, AM03, And06, MR09a, MR10a, MR10b], including Borel functions, $\mathbf{\Delta}^0_\alpha$-functions,[2] Lipschitz functions, uniformly continuous functions, functions of Baire class $< \alpha$ for a given additively closed countable ordinal α, $\mathbf{\Sigma}^1_n$-measurable functions, and so on. It turned out that all of them are very good when restricted to Borel sets, or even to larger collections of subsets of $^\omega\omega$ if suitable determinacy principles are assumed. In contrast, it is shown

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[1]Asking for no antichain at all seems unreasonable by the following considerations: let A be e.g. a proper open subset of a given Polish space X. On the one hand, checking membership in A cannot be considered strictly simpler or strictly more difficult than checking membership in its complement: this means that the degrees of A and $\neg A$ cannot be one strictly below the other in the hierarchy. On the other hand, the fact that open sets and closed sets have in general different (often complementary) combinatorial and topological properties, strongly suggests that the degrees of A and $\neg A$ should be kept distinct. Therefore such degrees must form an antichain of size 2.

[2]Given a countable ordinal $\alpha \geq 1$ and a Polish space X, a function $f \colon X \to X$ is called $\mathbf{\Delta}^0_\alpha$-*function* if $f^{-1}(A) \in \mathbf{\Sigma}^0_\alpha$ for every $A \in \mathbf{\Sigma}^0_\alpha$.

in [Her93, Her96, IST12, Sch12, MRSS12] that when considering the continuous reducibility on the real line $\mathbb{R}$ or, more generally, on arbitrary Polish spaces with *nonzero dimension*, then one usually gets a (very) bad hierarchy (and the same applies to some other classical kind of reducibilities, depending on the space under consideration).[3]

Given all these results, one may be tempted to conjecture that all "natural" $\mathcal{F}$-hierarchies on (Borel subsets of) a zero-dimensional Polish space X need to be very good. This conjecture is justified by the fact that every such space is homeomorphic to a closed subset (hence to a topological retract) of the Baire space, and a well-known transfer argument (see e.g. [MRSS12, Proposition 5.4]) shows that this already implies the following folklore result.

Proposition 1.2. *Let X be a zero-dimensional Polish space, and let $\mathcal{F}$ be an arbitrary reducibility on X which contains $\mathsf{W}(X)$, i.e. all continuous functions from X to itself. Then the $\mathcal{F}$-hierarchy $\mathsf{Deg}_{\boldsymbol{\Delta}^1_1}(\mathcal{F})$ on Borel subsets of X is very good.*

In fact, [MR09a, Theorem 3.1] (essentially) shows that this result can be further strengthened when X itself is a closed subset of ${}^\omega\omega$: if X is equipped with the restriction $\bar{d}_X$ of the canonical metric $\bar{d}$ on ${}^\omega\omega$, then $\mathsf{Deg}_{\boldsymbol{\Delta}^1_1}(\mathcal{F})$ is very good as soon as $\mathcal{F}$ contains the collection $\mathsf{L}(\bar{d}_X)$ of all $\bar{d}_X$-*nonexpansive* functions.

Despite the above mentioned results, in [MR12, Theorem 5.4, Proposition 5.10, and Theorem 5.11] it is shown that there are various natural reducibilities on ${}^\omega\omega$ that actually induce (very) bad hierarchies on its Borel subsets. In particular, it is shown that ${}^\omega\omega$ can be equipped with a complete ultrametric d', still compatible with its usual product topology, such that the $\mathcal{F}$-hierarchy on Borel (in fact, even just clopen) subsets of ${}^\omega\omega$ is very bad for $\mathcal{F}$ the collection of all the d'-nonexpansive (alternatively: d'-Lipschitz) functions.

Motivated by these results, in the present paper we continue this investigation by considering various complete ultrametrics on ${}^\omega\omega$ (compatible with its product topology) and, more generally, the collection of all *ultrametric Polish spaces* $X = (X, d)$, a very natural and interesting class which includes e.g. the space $\mathbb{Q}_p$ of p-adic numbers (for every prime $p \in \mathbb{N}$).[4] On such spaces, we then consider the hierarchies of degrees induced by one of the following reducibilities[5] on X:

- the collection $\mathsf{L}(d)$ of all nonexpansive functions, where $f\colon X \to X$ is called *nonexpansive* if $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$;
- the collection $\mathsf{Lip}(d)$ of all Lipschitz functions (with arbitrary constants), where $f\colon X \to X$ is a *Lipschitz function with constant L* (for a nonnegative real L) if $d(f(x), f(y)) \leq L \cdot d(x, y)$ for all $x, y \in X$;

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[3]Of course, one can further extend the class of topological spaces under consideration, and analyze e.g. the continuous reducibility on them: for example, [Sel05] considers the case of ω-algebraic domains (a class of spaces relevant in theoretical computer science), while [MRSS12] consider the broader class of the so-called quasi-Polish spaces. Moreover, it is possible to generalize the notion of reducibility itself by considering e.g. reducibilities between finite partitions (see e.g. [vEMS87, Her93, Sel05, Sel07, Sel10] and the references contained therein).

[4]More generally, the completion of any countable valued field K with valuation $|\cdot|_K\colon K \to \mathbb{R}$ and metric $d(x, y) = |x - y|_K$ (for $x, y \in K$) is always an ultrametric Polish space.

[5]Notice that since the metric topology on X is always zero-dimensional, it does not make much sense to consider reducibilities $\mathcal{F} \supseteq \mathsf{W}(X)$, because by Proposition 1.2 they always induce a very good hierarchy on Borel subsets of X.

- the collection $\mathsf{UCont}(d)$ of all uniformly continuous functions, where $f \colon X \to X$ is *uniformly continuous* if for every $\varepsilon \in \mathbb{R}^+$ there is a $\delta \in \mathbb{R}^+$ such that $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$ for all $x, y \in X$ (here $\mathbb{R}^+$ denotes the set of strictly positive reals).

The main results of the paper are the following:

(A) The $\mathsf{UCont}(d)$-hierarchy on Borel subsets of X is always very good (Theorem 3.10). Since by Proposition 3.4 it is possible to equip the Baire space with a compatible complete ultrametric d' such that $\mathsf{L}(\bar{d}) \not\subseteq \mathsf{UCont}(d')$ (where $\bar{d}$ is the usual metric on ${}^\omega\omega$), this also implies that $\mathsf{L}(\bar{d}) \subseteq \mathcal{F}$ is a sufficient but not necessary condition for the $\mathcal{F}$-hierarchy on Borel subsets of ${}^\omega\omega$ being very good (for $\mathcal{F}$ a reducibility on ${}^\omega\omega$).

(B) If X is perfect, then the $\mathsf{Lip}(d)$-hierarchy on the Borel subsets of X is either very good (if X has bounded diameter), or else it is very bad already when restricted to clopen subsets of X (if the diameter of X is unbounded). A technical strengthening of the property of having (un)bounded diameter (see Definition 3.11) works similarly for arbitrary ultrametric Polish spaces (Theorems 3.14 and 3.17, Corollary 3.19).

(C) If the range of d contains an honest increasing sequence (see Definition 4.1), then the $\mathsf{L}(d)$-hierarchy on clopen subsets of X is very bad (Theorem 4.2); in particular, this happens in the special case when X is perfect and has unbounded diameter. If instead the range of d is either finite or a decreasing ω-sequence converging to 0, then the $\mathsf{L}(d)$-hierarchy on Borel subsets of X is always very good (Theorem 4.7).

(D) It follows from the second part of (C) that if X is compact, then both[6] the $\mathsf{Lip}(d)$- and the $\mathsf{L}(d)$-hierarchy on Borel subsets of X are very good (Theorem 5.2).

(E) If we assume the Axiom of Choice AC, then the $\mathcal{F}$-hierarchy on (arbitrary subsets of) an uncountable X is very bad for every reducibility $\mathcal{F}$ such that $\mathsf{L}(d) \subseteq \mathcal{F} \subseteq \mathsf{Bor}(X)$, where $\mathsf{Bor}(X)$ is the collection of all Borel functions from X into itself (Theorem 6.3). If we further assume that $\mathsf{V} = \mathsf{L}$, then the $\mathcal{F}$-hierarchy on X is very bad already when restricted to $\boldsymbol{\Pi}^1_1$, i.e. coanalytic,[7] subsets of X (Theorem 6.11).

In particular, the results in (A)–(D) generalize those from [MR12, Section 5] and answer most of the questions in [MR12, Section 6]. Moreover, they allow us to construct discrete ultrametric Polish spaces $X = (X, d)$ whose $\mathsf{Lip}(d)$- and $\mathsf{L}(d)$-hierarchies are very bad (Corollaries 3.16 and 4.3), a fact which contradicts the conceivable conjecture that the $\mathsf{Lip}(d)$- and the $\mathsf{L}(d)$-hierarchy on them need to be (very) good since all subsets of such spaces are extremely simple (i.e. clopen). Notice also that the result mentioned in (E) under the assumption $\mathsf{V} = \mathsf{L}$ (which is best possible for most reducibilities $\mathcal{F}$ by Proposition 1.2 and the comment following it) can be viewed as an extension of the well-know classical result that if $\boldsymbol{\Pi}^1_1$-determinacy fails then there are proper $\boldsymbol{\Pi}^1_1$ sets which are not (Borel-)complete for coanalytic sets.

We end this introduction with two general remarks concerning the results presented in this paper:

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[6] Since on compact metric spaces continuity and uniform continuity coincide, the $\mathsf{UCont}(d)$-hierarchy on Borel subsets of a compact X is very good already by Proposition 1.2.

[7] Equivalently, to $\boldsymbol{\Sigma}^1_1$ (i.e. analytic) subsets of X.

i) to simplify the presentation, we will consider only $\mathcal{F}$-hierarchies on *Borel subsets* of a given ultrametric Polish space X (except in Section 6): this is because in this way we can avoid to assume any axiom beyond our basic theory $\mathsf{ZF}+\mathsf{DC}(\mathbb{R})$. However, as usual in Wadge theory, all our results can be extended to larger pointclasses $\mathbf{\Gamma} \subseteq \mathscr{P}(X)$ by assuming corresponding determinacy axioms (more precisely: the determinacy of subsets of $^{\omega}\omega$ which are Boolean combinations of sets in $\mathbf{\Gamma}$). In particular, under the full Axiom of Determinacy AD (asserting that all games on ω are determined), all these results remain true when considering *unrestricted $\mathcal{F}$-hierarchies* $\mathsf{Deg}(\mathcal{F})$ on X;

ii) when showing that a given $\mathcal{F}$-hierarchy on X (possibly restricted to some $\mathbf{\Gamma} \subseteq \mathscr{P}(X)$) is very bad, we will actually show that some very complicated partial (quasi-)order on $\mathscr{P}(\omega)$, like the inclusion relation $\subseteq$, or even the more complicated relation $\subseteq^*$ of inclusion modulo finite sets, embeds into such a hierarchy. This gives much stronger results, as it implies e.g. that the $\mathcal{F}$-hierarchy under consideration contains antichains of size the continuum and, in the case of $\subseteq^*$, that (under AC) every partial order of size $\aleph_1$ embeds into the $\mathcal{F}$-hierarchy on ($\mathbf{\Gamma}$-subsets of) X (see [Par63]).

2. Basic facts about ultrametric Polish spaces

Given a metric space $X = (X, d)$, we denote by τ_d the *metric topology (induced by d)*, i.e. the topology generated by the basic open balls $B_d(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$ (for some $x \in X$ and $\varepsilon \in \mathbb{R}^+$). When considered as a topological space, the space X is tacitly endowed with such topology, and therefore we will e.g. say that the metric space X is separable if there is a countable τ_d-dense subset of X, and similarly for all other topological notions. The diameter of X is *bounded* if there is $R \in \mathbb{R}^+$ such that $\sup\{d(x, y) \mid x, y \in X\} \leq R$, and *unbounded* otherwise.

A metric d on a space X is called *ultrametric* if it satisfies the following strengthening of the triangle inequality, for all $x, y, z \in X$:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

Definition 2.1. An *ultrametric Polish space* is a separable metric space $X = (X, d)$ such that d is a complete ultrametric. The collection of all ultrametric Polish spaces will be denoted by $\mathscr{X}$.

Every (τ_d-)closed subspace C of an ultrametric Polish space $X = (X, d)$ will be tacitly equipped with the metric $d_C = d \upharpoonright C$, which is obviously a complete ultrametric compatible with the relative topology on C induced by τ_d. When there is no danger of confusion, with a little abuse of notation the metric d_C will be sometimes denoted by d again.

Notation 2.2. Given an ultrametric Polish space $X = (X, d)$, we set $R(d) = \{d(x, y) \mid x, y \in X, x \neq y\}$, the set of all nonzero distances realized in X.

A typical example of an ultrametric Polish space is obtained by equipping the Baire space with the usual metric $\bar{d}$ defined by

$$\bar{d}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-n} & \text{if } n \text{ is smallest such that } x(n) \neq y(n) : \end{cases}$$

it is straightforward to check that $\bar{d}$ is actually an ultrametric generating the product topology on ${}^{\omega}\omega$, and obviously $R(\bar{d}) = \{2^{-n} \mid n \in \omega\}$. We will keep denoting this ultrametric by $\bar{d}$ throughout the paper.

We collect here some easy but useful facts about arbitrary ultrametric (Polish) spaces $X = (X, d)$:

(1) for every $x, y, z \in X$ two of the distances $d(x,y)$, $d(x,z)$, $d(y,z)$ are equal, and they are greater than or equal to the third (the "isosceles triangle" rule);

(2) for every $x, y, z \in X$, if $d(x,z) \neq d(y,z)$ then $d(x,y) = \max\{d(x,z), d(y,z)\}$. In particular, if $x, y, z, w \in X$ are such that $d(x,z), d(y,w) < d(x,y)$ then $d(z,w) = d(x,y)$;

(3) given a $(\tau_d$-)dense set $Q \subseteq X$, all distances are realized by elements of Q, that is: for every $x, y \in X$ there are $q, p \in Q$ such that $d(x,y) = d(q,p)$. In particular, if X is separable then $R(d)$ is countable;[8]

(4) for every $x \in X$ and $r \in \mathbb{R}^+$ the open ball $B_d(x,r)$ is actually clopen, and $B_d(y,r) = B_d(x,r)$ for every $y \in B_d(x,r)$. In particular, the topology τ_d is always zero-dimensional, and hence if X is an ultrametric Polish space, then it is homeomorphic to a closed subset of the Baire space by [Kec95, Theorem 7.8] (see also Lemma 3.5);

(5) given $x, y \in X$ and $r, s \in \mathbb{R}^+$, the (cl)open balls $B_d(x,r)$ and $B_d(y,s)$ are either disjoint, or else one of them contains the other.

To simplify the terminology, we adapt the definition of family of reducibilities introduced in [MRSS12, Definition 5.1] to the restricted context of ultrametric Polish spaces.

Definition 2.3. Let $\mathcal{F}$ be a collection of functions between ultrametric Polish spaces. For $X, Y \in \mathscr{X}$, denote by $\mathcal{F}(X,Y)$ the collection of all functions from $\mathcal{F}$ with domain X and range included in Y. The collection $\mathcal{F}$ is called *family of reducibilities* (on $\mathscr{X}$) if:

(1) it contains all the identity functions, i.e. $\mathrm{id}_X \in \mathcal{F}(X,X)$ for every $X \in \mathscr{X}$;

(2) it is closed under composition, i.e. for every $X, Y, Z \in \mathscr{X}$, $f \in \mathcal{F}(X,Y)$, and $g \in \mathcal{F}(Y,Z)$, the function $g \circ f$ belongs to $\mathcal{F}(X,Z)$;

Examples of family of reducibilities are the collections of all continuous functions, of all uniformly continuous functions, of all Lipschitz functions, and of all nonexpansive functions. Notice also that if $\mathcal{F}$ is a family of reducibilities then $\mathcal{F}(X) = \mathcal{F}(X,X)$ is a reducibility on the space X (for every $X \in \mathscr{X}$). The next simple lemma is a minor variation of [MRSS12, Proposition 5.4] and can be proved in a similar way.

Lemma 2.4. *Let $\mathcal{F}$ be a family of reducibilities and $X, Y \in \mathscr{X}$. Suppose that there is a surjective $f \in \mathcal{F}(X,Y)$ admitting a right inverse $g \in \mathcal{F}(Y,X)$. Then there is an embedding from $(\mathscr{P}(Y), \leq_{\mathcal{F}(Y)}, \neg)$ into $(\mathscr{P}(X), \leq_{\mathcal{F}(X)}, \neg)$.*

In particular, if $\mathcal{F}$ consists of Borel functions and the $\mathcal{F}(X)$-hierarchy on Borel subsets of X is (very) good, then also the $\mathcal{F}(Y)$-hierarchy on Borel subsets of Y is (very) good.

Proof. The map $\mathscr{P}(Y) \to \mathscr{P}(X) \colon A \mapsto f^{-1}(A)$ is the desired embedding. $\square$

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[8]Vice versa, for every countable $R \subseteq \mathbb{R}^+$ there is an ultrametric Polish space $X = (X,d)$ such that $R(d) = R$, for example $X = R \cup \{0\}$ with $d(x,y) = \max\{x,y\}$ for distinct $x, y \in X$.

3. Uniformly continuous and Lipschitz reducibilities

In [MR12, Question 6.2], it is asked whether one can equip the Baire space $^\omega\omega$ with a compatible complete ultrametric d' so that $\mathsf{L}(\bar{d}) \not\subseteq \mathsf{UCont}(d')$, and whether it is possible to strengthen this last condition to: the $\mathsf{UCont}(d')$-hierarchy on X is (very) bad. We start by answering positively the first part of this question.

Notation 3.1. Given a function $\phi\colon \omega \to \mathbb{R}^+$, we denote by $\mathrm{rg}(\phi)$ the *range of* ϕ, i.e. $\mathrm{rg}(\phi) = \{r \in \mathbb{R}^+ \mid \exists n \in \omega\,(\phi(n) = r)\}$.

Definition 3.2. Given a function $\phi\colon \omega \to \mathbb{R}^+$ with $\inf \mathrm{rg}(\phi) > 0$, define the metric d_ϕ on $^\omega\omega$ by setting for every $x, y \in {}^\omega\omega$

$$d_\phi(x, y) = \max\{\phi(x(0)), \phi(y(0))\} \cdot \bar{d}(x, y).$$

It is not hard to check that each d_ϕ is a complete ultrametric compatible with the product topology on $^\omega\omega$ (and that $\inf \mathrm{rg}(\phi) > 0$ is necessary for completeness).

Notation 3.3. Given a natural number $i \in \omega$ and an ordinal α, we denote by $i^{(\alpha)}$ the constant α-sequence with value i.

Proposition 3.4. *Let* $\phi\colon \omega \to \mathbb{R}^+\colon n \mapsto 2^n$. *Then* $\mathsf{L}(\bar{d}) \not\subseteq \mathsf{UCont}(d_\phi)$.

Proof. Consider the map $f\colon {}^\omega\omega \to {}^\omega\omega\colon n{}^\frown x \mapsto 3n{}^\frown x$. We show that for every $\varepsilon, \delta \in \mathbb{R}^+$ there are $x, y \in {}^\omega\omega$ such that $d_\phi(x, y) < \delta$ but $d_\phi(f(x), f(y)) > \varepsilon$. Let $0 \neq k \in \omega$ be such that $2^{-k} < \delta$. Then for every $n \geq k$ we get that setting $x = n^{(2n)}{}^\frown 0^{(\omega)}$ and $y = n^{(2n)}{}^\frown 1^{(\omega)}$,

$$d_\phi(x, y) = 2^n \cdot 2^{-2n} = 2^{-n} \leq 2^{-k} < \delta.$$

However,

$$d_\phi(f(x), f(y)) = 2^{3n} \cdot 2^{-2n} = 2^n,$$

hence letting n be large enough we get $d_\phi(f(x), f(y)) > \varepsilon$, as desired. $\qquad\square$

In order to answer the second half of [MR12, Question 6.2], we abstractly analyze the behavior of the $\mathsf{UCont}(d)$-hierarchy on an arbitrary ultrametric Polish space $X = (X, d)$. The following lemma uses standard arguments (see e.g. the proof of [Kec95, Theorem 7.8]), but we fully reprove it here for the reader's convenience.

Lemma 3.5. *Let* $X = (X, d)$ *be an ultrametric Polish space. Then there is a closed set* $C \subseteq {}^\omega\omega$ *and a bijection* $f\colon (C, \bar{d}) \to (X, d)$ *such that* f *is uniformly continuous and* f^{-1} *is nonexpansive. Moreover, if* X *has bounded diameter, then* f *is even Lipschitz, and if* X *has diameter* ≤ 1 *then we can alternatively require* f *to be nonexpansive and* f^{-1} *to be Lipschitz with constant* 2.

Proof. Let Q be a countable dense subset of X. Define the sets $A_s \subseteq X$ for $s \in {}^{<\omega}\omega$ recursively on $\mathrm{lh}(s)$ as follows: $A_\emptyset = X$. Given $A_s \subseteq X$, let $\{B_{s,i} \mid i < I\}$ (for some $I \leq \omega$) be an enumeration without repetitions of the set of open balls $\{B_d(x, 2^{-\mathrm{lh}(s)}) \mid x \in Q \cap A_s\}$, and set $A_{s{}^\frown i} = B_{s,i}$ if $i < I$ and $A_{s{}^\frown i} = \emptyset$ otherwise. Since d is an ultrametric, one can easily check that the family $(A_s)_{s \in {}^{<\omega}\omega}$ is a Luzin scheme with vanishing diameter consisting of clopen sets, and with the further property that $A_s = \bigcup_{n \in \omega} A_{s{}^\frown n}$ for every $s \in {}^{<\omega}\omega$. Therefore the set $C = \{x \in {}^\omega\omega \mid \bigcap_{n \in \omega} A_{x \restriction n} \neq \emptyset\}$ is a closed subset of $^\omega\omega$, and the map $f\colon C \to X$ sending $x \in C$ to the unique element in $\bigcap_{n \in \omega} A_{x \restriction n}$ is a bijection. So it remains only to check that such f has the desired properties.

Given $\varepsilon > 0$, let $n \in \omega$ be smallest such that $2^{-n} \leq \varepsilon$, and set $\delta = 2^{-n}$. If $x, y \in C$ are such that $\bar{d}(x, y) < \delta$, then $x \upharpoonright (n+1) = y \upharpoonright (n+1)$, which implies $f(x), f(y) \in A_{x \upharpoonright (n+1)}$. By definition of the A_s, this implies that $d(f(x), f(y)) < 2^{-n} \leq \varepsilon$. This shows that f is uniformly continuous.

Further assuming that X be of bounded diameter, we get that f is Lipschitz with constant $\max\{2, k\}$, where $k \in \omega$ is an arbitrary bound to the diameter of X, i.e. it is such that $d(x, y) \leq k$ for every $x, y \in X$. To see this, fix distinct $x, y \in C$. If $x(0) \neq y(0)$ then $d(f(x), f(y)) \leq k \leq k \cdot \bar{d}(x, y)$ by our choice of $k \in \omega$. Let now $n \neq 0$ be smallest such that $x(n) \neq y(n)$, so that $\bar{d}(x, y) = 2^{-n}$. Since $x \upharpoonright n = y \upharpoonright n$ we get that $f(x), f(y) \in A_{x \upharpoonright n}$, which implies $d(f(x), f(y)) < 2^{-(n-1)}$: therefore $d(f(x), f(y)) < 2 \cdot 2^{-n} = 2 \cdot d(x, y)$.

Now fix $x, y \in C$, and let $n \in \omega$ be such that $\bar{d}(x, y) = 2^{-n}$. Since $x(n) \neq y(n)$ implies $A_{x \upharpoonright (n+1)} \cap A_{y \upharpoonright (n+1)} = \emptyset$, we get that $d(f(x), f(y)) \geq 2^{-n}$ (because d is an ultrametric), and hence $\bar{d}(x, y) \leq d(f(x), f(y))$. This shows that f^{-1} is nonexpansive.

Finally, assume that X has diameter ≤ 1. In the construction above, redefine the collections $\{B_{s,i} \mid i < I\}$ as enumerations without repetitions of $\{B_d(x, 2^{-(\mathrm{lh}(s)+1)}) \mid x \in Q \cap A_s\}$, and then use this new sets to define the A_s's and the map f. Arguing as before, one can easily check that f is now nonexpansive while f^{-1} is Lipschitz with constant 2, as required. $\qquad\square$

Remark 3.6. The special case of Lemma 3.5 where X has diameter ≤ 1 already appeared (with the same proof) in [MR09b, Theorem 4.1]. However, such a result cannot be literally extended to an arbitrary ultrametric Polish space X, and in fact the assumptions in Lemma 3.5 are optimal. To see this, note that if X has unbounded diameter then we cannot require a map f as in Lemma 3.5 to be Lipschitz because every Lipschitz image of a space with bounded diameter (like any set $C \subseteq {}^{\omega}\omega$) has necessarily bounded diameter too. Similarly, a nonexpansive image of a set of diameter $\leq R$ (for some $R \in \mathbb{R}^+$), has diameter $\leq R$ too.

Definition 3.7. Let X be a topological space, $\mathcal{F}$ be a collection of functions from X to itself, and $A \subseteq X$. We call $\mathcal{F}$-*retraction* of X onto A any surjection $f \in \mathcal{F}$ from X onto A such that $f \upharpoonright A = \mathrm{id}_A$; if such a function exists we also say that A is an $\mathcal{F}$-*retract* of X.

Recall from [Kec95, Proposition 2.8] that if $\emptyset \neq A \subseteq C$ are closed subsets of the Baire space, then there is an $\mathsf{L}(\bar{d}_C)$-retraction (i.e. a nonexpansive retraction) of C onto A — a fact that will be repeatedly used throughout the paper. The next corollary generalizes this result to arbitrary ultrametric Polish spaces, provided that we slightly weaken the requirement that the retraction be nonexpansive.

Corollary 3.8. *Let $X = (X, d)$ be an ultrametric Polish space. For every nonempty closed $A \subseteq X$, there is a uniformly continuous retraction $r \colon X \twoheadrightarrow A$. If moreover A has bounded diameter, then the retraction r can be taken to be Lipschitz.*

Proof. Let C and f be as in Lemma 3.5, with f uniformly continuous and f^{-1} nonexpansive. Notice that since f is, in particular, a homeomorphism, the set $A' = f^{-1}(A)$ is a nonempty closed subset of C. Let $g \colon C \to A'$ be a nonexpansive retraction: then $r = f \circ g \circ f^{-1} \colon (X, d) \twoheadrightarrow (A, d)$ is the desired uniformly continuous retraction.

Assume now that A has bounded diameter, and let C, f, and g be as in the previous paragraph. Arguing as in the proof of Lemma 3.5, one can easily check that $f \restriction A' \colon (A', \bar{d}) \to (X, d)$ is actually Lipschitz (since A has bounded diameter): therefore $r = (f \restriction A') \circ g \circ f^{-1} \colon (X, d) \twoheadrightarrow (A, d)$ is the desired Lipschitz retraction. $\square$

Remark 3.9. It is not possible in general to strengthen Corollary 3.8 by requiring the reduction to be nonexpansive, even if we require the entire X to have small diameter. To see this, let $X = \{0\} \cup \left\{ \frac{1}{2} + 2^{-(n+1)} \mid n \in \omega \right\}$, and set $d(x, y) = \max\{x, y\}$ for all distinct $x, y \in X$. Then $X = (X, d)$ is a discrete ultrametric Polish space of diameter ≤ 1. Consider the clopen set $A = X \setminus \{0\}$, and let $f \colon X \twoheadrightarrow A$ be a retraction. Let $n \in \omega$ be such that $f(0) = \frac{1}{2} + 2^{-(n+1)}$: then setting $x = 0$ and $y = \frac{1}{2} + 2^{-(n+2)}$ we get that

$$d(f(x), f(y)) = d(f(x), y) = \frac{1}{2} + 2^{-(n+1)} > \frac{1}{2} + 2^{-(n+2)} = d(x, y),$$

so f is expansive.

Theorem 3.10. *The $\mathsf{UCont}(d)$-hierarchy $\mathsf{Deg}_{\mathbf{\Delta}^1_1}(\mathsf{UCont}(d))$ on the Borel subsets of an arbitrary ultrametric Polish space $X = (X, d)$ is always very good. If X has bounded diameter, then the $\mathsf{Lip}(d)$-hierarchy $\mathsf{Deg}_{\mathbf{\Delta}^1_1}(\mathsf{Lip}(d))$ on the Borel subsets of X is very good as well.*

Proof. Let $C \subseteq {}^\omega\omega$ and $f \colon C \to X$ be as in Lemma 3.5, and let $g \colon ({}^\omega\omega, \bar{d}) \to (C, \bar{d})$ be a nonexpansive retraction. Then f^{-1} is a right inverse of $g \circ f$, and hence the result follows from Lemma 2.4 and the fact that both the $\mathsf{UCont}(\bar{d})$-hierarchy and the $\mathsf{Lip}(\bar{d})$-hierarchy are very good by [MR10a]. $\square$

In particular, this fully answers in the negative the second half of [MR12, Question 6.2]. Moreover, Theorem 3.10 provides also a negative answer to [MR12, Question 6.1]: letting ϕ be as in Proposition 3.4, we get that the set $\mathsf{UCont}(d_\phi)$ of uniformly continuous functions is a surjective image of ${}^\omega\omega$,[9] it does not contain $\mathsf{L}(\bar{d})$, but it induces a very good hierarchy on the Borel subsets (or, further assuming AD, on the collection of all subsets) of ${}^\omega\omega$.

Theorem 3.10 shows that having a bounded diameter is a sufficient condition for having that the $\mathsf{Lip}(d)$-hierarchy on the Borel subsets of an ultrametric Polish space $X = (X, d)$ is very good. In fact, we are now going to show that a technical strengthening of this condition is both necessary and sufficient for that.

Definition 3.11. Let $X = (X, d)$ be an (ultra)metric Polish space. We say that the diameter of X is *nontrivially unbounded* if for every $k \in \omega$ and every $\varepsilon \in \mathbb{R}^+$ there are $x, y \in X$ with $d(x, y) > k$ such that both x and y are not ε-isolated.[10]

Notice that if X is *perfect*, then the diameter of X is nontrivially unbounded if and only if it is unbounded.

---

[9]When working in models of AD (as it is often the case when dealing with Wadge-like hierarchies), for technical reasons it is often preferable to express "cardinality inequality" using surjections instead of injections. Therefore the stated property should be intended (in any model of ZF) as: the cardinality of $\mathsf{UCont}(d_\phi)$ is not larger than that of the Baire space. Obviously, further assuming the Axiom of Choice AC this just means that $\mathsf{UCont}(d_\phi)$ has cardinality $\leq 2^{\aleph_0}$.

[10]Recall that a point x of a metric space is called *ε-isolated* (for some $\varepsilon \in \mathbb{R}^+$) if $B_d(x, \varepsilon) = \{x\}$.

Example 3.12. Let p be a prime natural number, and let $\mathbb{Q}_p$ be the ultrametric Polish space of p-adic numbers equipped with the usual p-adic metric d_p: then $\mathbb{Q}_p$ has unbounded diameter and is perfect (hence its diameter is nontrivially unbounded). To see the former, given $k \in \omega$ let $n \in \omega$ be such that $n \geq 2$ and $k < p^n$: setting $x = p^{-1}$ and $y = p^{-n}$ we easily get $d_p(x,y) = p^n > k$. To see that $\mathbb{Q}_p$ is also perfect, fix an arbitrary $q \in \mathbb{Q}$, and given $\varepsilon \in \mathbb{R}^+$ let $l \in \omega$ be such that $p^{-l} < \varepsilon$: then $q' = q - p^l$ is distinct from q and $d_p(q,q') = p^{-l} < \varepsilon$. This shows that q is not isolated, and since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$ we are done.

Notation 3.13. We let $\subseteq^*$ denote the relation of inclusion modulo finite sets between subsets of ω, i.e. for every $a, b \subseteq \omega$ we set

$$a \subseteq^* b \iff \exists \bar{k} \in \omega \, \forall k \geq \bar{k} \, (k \in a \Rightarrow k \in b).$$

Theorem 3.14. *Let $X = (X,d)$ be an ultrametric Polish space, and assume that its diameter is nontrivially unbounded. Then there is a map ψ from $\mathscr{P}(\omega)$ into the clopen subsets of X such that for all $a, b \subseteq \omega$:*

(1) if $a \subseteq^* b$ then $\psi(a) \leq_{\mathsf{L}(d)} \psi(b)$;
(2) if $\psi(a) \leq_{\mathsf{Lip}(d)} \psi(b)$ then $a \subseteq^* b$.

In particular, $(\mathscr{P}(\omega), \subseteq^*)$ embeds into both $\mathsf{Deg}_{\mathbf{\Delta}^0_1}(\mathsf{Lip}(d))$ and $\mathsf{Deg}_{\mathbf{\Delta}^0_1}(\mathsf{L}(d))$.

Proof. Let $(q_n)_{n \in \omega}$ be an enumeration of a countable dense subset Q of X. We first recursively construct two sequences $(r_n)_{n \in \omega}$, $(s_n)_{n \in \omega}$ of nonnegative reals and two sequences $(x_n)_{n \in \omega}$, $(y_n)_{n \in \omega}$ of points of X such that for all distinct $n, m \in \omega$ the following properties hold:

(a) $d(x_n, x_m) = r_{\max\{n,m\}}$ and $d(x_n, y_n) = s_n$;
(b) $r_{n+1} > \max\{n + 1, r_n^2\}$ (in particular, $(r_n)_{n \in \omega}$ is strictly increasing and unbounded in $\mathbb{R}^+$);
(c) $s_0 < 1$ and $s_{n+1} < \frac{s_n}{r_n + 1}$ (in particular, $(s_n)_{n \in \omega}$ is a strictly decreasing sequence).

Claim 3.14.1. *If $x \in X$ is not ε-isolated then there are at least two distinct $q_i, q_j \in Q$ such that $q_i, q_j \in B_d(x, \varepsilon)$.*

Proof of the Claim. Since x is not ε-isolated, there is $y \in B_d(x, \varepsilon)$ such that $x \neq y$. By density of Q, there are $q_i, q_j \in Q$ such that $q_i \in B_d(x, d(x,y))$ and $q_j \in B_d(y, d(x,y))$. Then $q_i \neq q_j$ since $B_d(x, d(x,y)) \cap B_d(y, d(x,y)) = \emptyset$, while $q_i, q_j \in B_d(x, \varepsilon)$ because $B_d(x, d(x,y)), B_d(y, d(x,y)) \subseteq B_d(x, \varepsilon)$ by $d(x,y) < \varepsilon$. $\qquad\square$

Let $x \in X$ be not 1-isolated (such an x exists because the diameter of X is nontrivially unbounded), and let q_i, q_j be as in Claim 3.14.1 for $\varepsilon = 1$. Then we set $x_0 = q_i$, $y_0 = q_j$, $r_0 = 0$, and $s_0 = d(q_i, q_j)$. Now assume that x_n, y_n, r_n, and s_n have been defined. Let $x, y \in X$ be such that $d(x,y) > \max\{n + 1, r_n^2\}$ and x, y are not $\frac{s_n}{r_n+1}$-isolated. Then at least one of x and y has distance greater than $\max\{n + 1, r_n^2\}$ from x_n (and hence also from all the x_m for $m \leq n$): if not, then we would have $d(x,y) \leq \max\{d(x, x_n), d(y, x_n)\} \leq \max\{n + 1, r_n^2\}$, contradicting our choice of x, y. So we may assume without loss of generality that $d(x, x_n) > \max\{n + 1, r_n^2\}$ and x is not $\frac{s_n}{r_n+1}$-isolated. Let q_i, q_j be as in Claim 3.14.1 for $\varepsilon = \frac{s_n}{r_n+1}$, and set $x_{n+1} = q_i$, $y_{n+1} = q_j$, $r_{n+1} = d(q_i, x_n)$, and $s_{n+1} = d(q_i, q_j)$. Since $d(q_i, x) < \frac{s_n}{r_n+1} \leq 1 \leq \max\{n + 1, r_n^2\}$, we have $r_{n+1} = d(q_i, x_n) = d(x, x_n) > \max\{n + 1, r_n^2\}$. Moreover, $s_{n+1} < \frac{s_n}{r_n+1}$ by the fact

that $q_i, q_j \in B_d(x, \frac{s_n}{r_n + 1})$. Arguing by induction on $n \in \omega$, it is then easy to check that the sequences constructed in this way have all the desired properties.

Given $a \subseteq \omega$, let $\hat{a} = \{2i \mid i \in \omega\} \cup \{2i + 1 \mid i \in a\}$, so that $\hat{a}$ is always infinite and for every $a, b \subseteq \omega$

$$a \subseteq^* b \iff \hat{a} \subseteq^* \hat{b}.$$

For $a \subseteq \omega$, set $\psi(a) = \bigcup_{i \in \hat{a}} B_d(x_i, s_i)$. Clearly, each $\psi(a)$ is an open subset of X. To see that it is also closed, observe that $B_d(x_i, s_i) \subseteq B_d(x_i, 1)$ for every $i \in \omega$ by our choice of the s_i's, and that for distinct $i, j \in \omega$ the clopen balls $B_d(x_i, 1)$ and $B_d(x_j, 1)$ are disjoint by our choice of the x_i's and of the r_i's: therefore, since the open balls in X are automatically closed we get that

$$X \setminus \psi(a) = \bigcup \left\{ B_d(z, 1) \mid z \notin \bigcup_{i \in \hat{a}} B_d(x_i, 1) \right\} \cup \bigcup \{ B_d(x_i, 1) \setminus B_d(x_i, s_i) \mid i \in \hat{a} \}$$

is open.

Let now $a, b \subseteq \omega$ be such that $a \subseteq^* b$, which in particular implies $\hat{a} \subseteq^* \hat{b}$, and let $0 \neq \bar{k} \in \omega$ be such that $\bar{k} \in \hat{a}$ and $k \in \hat{a} \Rightarrow k \in \hat{b}$ for every $k \geq \bar{k}$. Define $f \colon (X, d) \to (X, d)$ as follows:

$$f(x) = \begin{cases} x_{\bar{k}} & \text{if } x \in B_d(x_i, s_i), i < \bar{k}, i \in \hat{a} \\ y_{\bar{k}} & \text{if } x \in B_d(x_0, r_{\bar{k}}) \setminus \bigcup\{B_d(x_i, s_i) \mid i < \bar{k}, i \in \hat{a}\} \\ y_i & \text{if } x \in B_d(x_i, s_i), i \geq \bar{k}, i \notin \hat{a} \\ x & \text{otherwise.} \end{cases}$$

It is straightforward to check that f reduces $\psi(a)$ to $\psi(b)$, so we only need to check that f is nonexpansive, and this amounts to check that if x, y are distinct points of X which fall in different cases in the definition of f, then $d(f(x), f(y)) \leq d(x, y)$. A careful inspection shows that the unique nontrivial cases are the following:

case A: $x \in B_d(x_0, r_{\bar{k}})$, while $y \notin B_d(x_0, r_{\bar{k}}) \cup \bigcup\{B_d(x_i, s_i) \mid i \geq \bar{k}, i \notin \hat{a}\}$. Then $d(x, y) \geq r_{\bar{k}}$ (by case assumption) and $d(x, f(x)) = r_{\bar{k}}$ (because either $f(x) = x_{\bar{k}}$ or $f(x) = y_{\bar{k}}$, depending on whether $x \in B_d(x_i, s_i)$ for some $i \in \hat{a}$ smaller than $\bar{k}$ or not). Since in the case under consideration $f(y) = y$, we get that either $d(f(x), f(y)) \leq r_{\bar{k}}$, or else $d(f(x), f(y)) = d(f(x), y) = d(x, y)$ by the isosceles triangle rule: in both cases, $d(f(x), f(y)) \leq d(x, y)$ as required.

case B: $x \in B_d(x_0, r_{\bar{k}}) \setminus \bigcup\{B_d(x_i, s_i) \mid i < \bar{k}, i \in \hat{a}\}$, while $y \in B_d(x_i, s_i)$ for some $i \geq \bar{k}$, $i \notin \hat{a}$. Then since $d(x, x_0) < r_{\bar{k}}$ and $d(x_0, y) = r_i \geq r_{\bar{k}}$, we get $d(x, y) = r_i$. Since by case assumption $f(x) = y_{\bar{k}}$ and $f(y) = y_i$, either $f(x) = f(y)$ (in case $i = \bar{k}$) or $d(f(x), f(y)) = r_i$, and hence we again get $d(f(x), f(y)) \leq d(x, y)$, as required.

case C: $x \in B_d(x_i, s_i)$ for some $i \geq \bar{k}$, $i \notin \hat{a}$, while $y \notin B_d(x_0, r_{\bar{k}}) \cup \bigcup\{B_d(x_i, s_i) \mid i \geq \bar{k}, i \notin \hat{a}\}$. Then $d(x, y) \geq s_i$, $d(x, f(x)) = s_i$ (because $f(x) = y_i$), and $f(y) = y$: this implies that either $d(f(x), f(y)) \leq s_i$ or $d(f(x), f(y)) = d(f(x), y) = d(x, y)$, so that in any case $d(f(x), f(y)) \leq d(x, y)$.

This concludes the proof of part (1).

We now prove part (2) of the theorem. Given $a, b \subseteq \omega$, assume that $f \colon (X, d) \to (X, d)$ is a $\mathsf{Lip}(d)$-reduction of $\psi(a)$ to $\psi(b)$, and let $0 \neq n \in \omega$ be such that $d(f(x), f(y)) \leq r_n \cdot d(x, y)$ for every $x, y \in X$ (such an n exists because $(r_n)_{n \in \omega}$ is unbounded in $\mathbb{R}^+$ by (b) above). Notice that, necessarily, $f\left(\bigcup\{B_d(x_i, s_i) \mid i \in \hat{a}\}\right) =$

$f(\psi(a)) \subseteq \psi(b) \subseteq \bigcup_{j \in \omega} B_d(x_j, s_j)$. We now argue as in the proof of [MR12, Theorem 5.4].

Claim 3.14.2. *Fix an arbitrary $i \in \hat{a}$. If there are $x \in B_d(x_i, s_i)$ and $j \geq n$ such that $f(x) \in B_d(x_j, s_j)$, then $f(B_d(x_i, s_i)) \subseteq B_d(x_j, s_j)$.*

Proof of the Claim. Suppose not, and let $y \in B_d(x_i, s_i)$ and $j' \neq j$ be such that $f(y) \in B_d(x_{j'}, s_{j'})$. Then

$$d(f(x), f(y)) = \max\{r_j, r_{j'}\} \geq r_j \geq r_n \cdot 1 > r_n \cdot s_i > r_n \cdot d(x, y),$$

contradicting the choice of n. $\qquad\square$

Claim 3.14.3. *For every $i \in \hat{a}$ such that $i > n$, $f(B_d(x_i, s_i)) \subseteq B_d(x_j, s_j)$ for some $j \geq i$.*

Proof. Suppose towards a contradiction that there are $x \in B_d(x_i, s_i)$ and $j < i$ such that $f(x) \in B_d(x_j, s_j)$, so that, in particular, $j \in \hat{b}$ because $x \in \psi(a)$ and f reduces $\psi(a)$ to $\psi(b)$. Then since $d(x, y_i) = s_i$, by our choice of the s_i's we get

$$d(f(x), f(y_i)) \leq r_n \cdot s_i \leq r_{i-1} \cdot s_i < s_{i-1} \leq s_j,$$

and hence $f(y_i) \in B_d(f(x), s_j) = B_d(x_j, s_j) \subseteq \psi(b)$: but this contradicts the fact that f is a reduction of $\psi(a)$ to $\psi(b)$, because $y_i \notin \psi(a)$ while $B_d(x_j, s_j) \subseteq \psi(b)$ since $j \in \hat{b}$. Thus, given an arbitrary $x \in B_d(x_i, s_i)$ there is $j \geq i > n$ such that $f(x) \in B_d(x_j, s_j)$: by Claim 3.14.2, we then get $f(B_d(x_i, s_i)) \subseteq B_d(x_j, s_j)$, as required. $\qquad\square$

Let now $\bar{\imath}$ be the smallest element of $\hat{a}$. By Claim 3.14.2, either $f(B_d(x_{\bar{\imath}}, s_{\bar{\imath}})) \subseteq \bigcup_{j < n} B_d(x_j, s_j)$, or $f(B_d(x_{\bar{\imath}} s_{\bar{\imath}})) \subseteq B_d(x_j, s_j)$ for some $j \geq n$. Therefore, in both cases there is $\bar{k} > \max\{n, \bar{\imath}\}$ such that $f(B_d(x_{\bar{\imath}}, s_{\bar{\imath}})) \subseteq \bigcup_{j \leq \bar{k}} B_d(x_j, s_j)$: we claim that $k \in \hat{a} \Rightarrow k \in \hat{b}$ for every $k \geq \bar{k}$, which also implies $a \subseteq^* b$.

Fix $k \geq \bar{k}$ such that $k \in \hat{a}$. By Claim 3.14.3 and $\bar{k} > n$, there is $j \geq k$ such that $f(B_d(x_k, s_k)) \subseteq B_d(x_j, s_j)$. Assume towards a contradiction that $j > k$: then

$$d(f(x_{\bar{\imath}}), f(x_k)) = r_j > r_k \cdot r_k > r_n \cdot r_k = r_n \cdot d(x_{\bar{\imath}}, x_k),$$

contradicting the choice of n. Therefore $f(B_d(x_k, s_k)) \subseteq B_d(x_k, s_k)$, which in particular implies that $\psi(b) \cap B_d(x_k, s_k) \neq \emptyset$ (since $x_k \in \psi(a)$ and f reduces $\psi(a)$ to $\psi(b)$): but this means that $k \in \hat{b}$, and hence we are done. $\qquad\square$

Applying Theorem 3.14 to the space $\mathbb{Q}_p$ of p-adic numbers (which is possible by Example 3.12) we get the following corollary.

Corollary 3.15. *Let p be a prime natural number, and let d_p be the p-adic metric on the space $\mathbb{Q}_p$. Then both the $\mathsf{Lip}(d_p)$- and the $\mathsf{L}(d_p)$-hierarchies are very bad already when restricted to clopen subsets of $\mathbb{Q}_p$.*

The condition on the diameter of $X = (X, d)$ used to prove Theorem 3.14 is very weak: this allows us to construct extremely simple (in fact: discrete) ultrametric Polish spaces $X = (X, d)$ with the property that their $\mathsf{Lip}(d)$- and $\mathsf{L}(d)$-hierarchies are both very bad, despite the fact that all their subsets are topologically simple (i.e. clopen).

Corollary 3.16. *There exists a discrete (hence countable) ultrametric Polish space $X_0 = (X_0, d_0)$ such that $(\mathscr{P}(\omega), \subseteq^*)$ embeds into both the $\mathsf{Lip}(d_0)$- and the $\mathsf{L}(d_0)$-hierarchy on (the clopen subsets of) X_0. In particular, $\mathsf{Deg}(\mathsf{Lip}(d_0)) = \mathsf{Deg}_{\mathbf{\Delta}^0_1}(\mathsf{Lip}(d_0))$ and $\mathsf{Deg}(\mathsf{L}(d_0)) = \mathsf{Deg}_{\mathbf{\Delta}^0_1}(\mathsf{L}(d_0))$ are both very bad.*

Proof. Let $X_0 = \{x_n^i \mid n \in \omega, i = 0, 1\}$ and set

$$
d_0(x_n^i, x_m^j) = \begin{cases} 0 & \text{if } n = m \text{ and } i = j \\ 2^{-n} & \text{if } n = m \text{ and } i \neq j \\ \max\{n, m\} & \text{if } n \neq m. \end{cases}
$$

It is easy to check that $X_0 = (X_0, d_0)$ is a discrete ultrametric Polish space. Now observe that the diameter of X_0 is nontrivially unbounded. In fact, given $n \in \omega$ and $\varepsilon \in \mathbb{R}^+$, let k be minimal such that $2^{-k} < \varepsilon$ and $l = \max\{n, k\}$: then $d_0(x_l^0, x_{l+1}^0) = l + 1 > n$, and the points x_l^1 and x_{l+1}^1 witness that x_l^0 and x_{l+1}^0 are not ε-isolated. Therefore X_0 is as desired by Theorem 3.14. $\qquad\square$

The next proposition extends Theorem 3.10 and shows that the condition on X in Theorem 3.14 is optimal.

Theorem 3.17. *Let $X = (X, d)$ be an ultrametric Polish space whose diameter is not nontrivially unbounded. Then the $\mathsf{Lip}(d)$-hierarchy $\mathsf{Deg}_{\mathbf{\Delta}^1_1}(\mathsf{Lip}(d))$ on Borel subsets of X is very good.*

Proof. Let $n \in \omega$ and $\varepsilon \in \mathbb{R}^+$ be such that for every x, y, if $d(x, y) > n$ then at least one of x and y is ε-isolated.

Let us first consider the degenerate case in which all points of X are ε-isolated. Since constant functions are always (trivially) Lipschitz, we get that the sets X and $\emptyset$ are $\mathsf{Lip}(d)$-incomparable, and that they are both (strictly) $\leq_{\mathsf{Lip}(d)}$-below any other set $\emptyset, X \neq A \subseteq X$. Assume now that $B \subseteq X$ is another set which is different from both $\emptyset$ and X: we claim that then $A \equiv_{\mathsf{Lip}(d)} B$. To see this, fix $\bar{x} \in B$ and $\bar{y} \in \neg B$, and for every $x \in X$ set $f(x) = \bar{x}$ if $x \in A$ and $f(x) = \bar{y}$ if $x \in \neg A$. Then $f \colon (X, d) \to (X, d)$ reduces A to B. Moreover, since for all distinct $x, y \in X$ we have $d(x, y) \geq \varepsilon$ (because both x and y are ε-isolated), we get

$$
d(f(x), f(y)) \leq d(\bar{x}, \bar{y}) = \frac{d(\bar{x}, \bar{y})}{\varepsilon} \cdot \varepsilon \leq \frac{d(\bar{x}, \bar{y})}{\varepsilon} \cdot d(x, y),
$$

so that f is Lipschitz with constant $\frac{d(\bar{x}, \bar{y})}{\varepsilon}$. This shows that $A \leq_{\mathsf{Lip}(d)} B$. Switching the role of A and B, we get that also $B \leq_{\mathsf{Lip}(d)} A$, and hence we are done. Therefore we have shown that the $\mathsf{Lip}(d)$-hierarchy on X is constituted by the two $\mathsf{Lip}(d)$-incomparable degrees $[\emptyset]_{\mathsf{Lip}(d)} = \{\emptyset\}$ and $[X]_{\mathsf{Lip}(d)} = \{X\}$, plus a unique $\mathsf{Lip}(d)$-degree above them containing all other subsets of X, and is thus (trivially) very good.

Assume now that there is a non-ε-isolated point $x_0 \in X$, and set $X' = B_d(x_0, n+1)$. By our choice of n and ε, we get that $d(x, y) \geq n + 1$ for every $x \in X'$ and $y \in X \setminus X'$, and that each $y \in X \setminus X'$ is ε-isolated (because $d(x_0, y) > n$ and x_0 is not ε-isolated). We first prove the following useful claim.

Claim 3.17.1. *Let $A, B \subseteq X$ be such that $B \neq \emptyset, X$. If there is a Lipschitz reduction $f \colon (X', d_{X'}) \to (X', d_{X'})$ of $A' = A \cap X'$ to $B' = B \cap X'$, then $A \leq_{\mathsf{Lip}(d)} B$.*

Proof. Let f be as in the hypothesis of the claim, and let $1 \leq k \in \omega$ be such that $d(f(x), f(y)) \leq k \cdot d(x, y)$ for every $x, y \in X'$. Fix $\bar{x} \in B$ and $\bar{y} \in \neg B$, and extend f to the map $\hat{f} \colon (X, d) \to (X, d)$ by letting $\hat{f}(x) = \bar{x}$ if $x \in A \setminus X'$ and $\hat{f}(x) = \bar{y}$ if $x \in X \setminus (X' \cup A)$. Clearly, $\hat{f}$ reduces A to B, and we claim that $\hat{f}$ is Lipschitz with constant c, where c is

$$c = \max \left\{ k, \frac{d(\bar{x}, \bar{y})}{\varepsilon}, \frac{d(x_0, \bar{x})}{n+1}, \frac{d(x_0, \bar{y})}{n+1} \right\}.$$

Fix arbitrary $x, y \in X$. If $x, y \in X'$, then

$$d(\hat{f}(x), \hat{f}(y)) = d(f(x), f(y)) \leq k \cdot d(x, y) \leq c \cdot d(x, y)$$

by our choice of $k \in \omega$. If $x, y \in X \setminus X'$, then $d(x, y) \geq \varepsilon$ because both x and y are ε-isolated, and either $\hat{f}(x) = \hat{f}(y)$ or $d(\hat{f}(x), \hat{f}(y)) = d(\bar{x}, \bar{y})$. Therefore in both cases

$$d(\hat{f}(x), \hat{f}(y)) \leq \frac{d(\bar{x}, \bar{y})}{\varepsilon} \cdot \varepsilon \leq c \cdot d(x, y).$$

Let now $x \in X'$ and $y \in X \setminus X'$, and assume without loss of generality that $\hat{f}(y) = \bar{x}$ (the case $\hat{f}(y) = \bar{y}$ is analogous, just systematically replace $\bar{x}$ with $\bar{y}$ in the argument below). Then either $\bar{x} \in X'$, in which case $d(\hat{f}(x), \hat{f}(y)) < n+1 \leq d(x, y) \leq c \cdot d(x, y)$ (since $c \geq k \geq 1$), or else

$$d(\hat{f}(x), \hat{f}(y)) = d(x_0, \bar{x}) = \frac{d(x_0, \bar{x})}{n+1} \cdot n + 1 \leq c \cdot d(x, y).$$

The case $x \in X \setminus X'$ and $y \in X'$ can be treated similarly, so in all cases we obtained $d(\hat{f}(x), \hat{f}(y)) \leq c \cdot d(x, y)$, as required. $\qquad\square$

We now want to show that the $\mathsf{SLO}^{\mathsf{Lip}(d)}$ principle holds for Borel subsets of X, so let us fix arbitrary Borel $A, B \subseteq X$. Assume first that $B = X$. Then either $A = X$, in which case the identity map on X witnesses $A \leq_{\mathsf{Lip}(d)} B$, or else $\neg A \neq \emptyset$, in which case any constant map with value $\bar{x} \in \neg A$ witnesses $B \leq_{\mathsf{Lip}(d)} \neg A$. The symmetric case $B = \emptyset$ can be dealt with in a similar way, so in what follows we can assume without loss of generality that $B \neq \emptyset, X$. Moreover, switching the role of A and B in the argument above we may further assume that $A \neq \emptyset, X$. Set $A' = A \cap X'$ and $B' = B \cap X'$. Since X' has bounded diameter, by Theorem 3.10 there is a Lipschitz function $f \colon (X', d) \to (X', d)$ such that either $f^{-1}(B') = A'$ or $f^{-1}(X' \setminus A') = B'$. Since $\neg A \cap X' = X' \setminus A'$, applying Claim 3.17.1 we get that either $A \leq_{\mathsf{Lip}(d)} B$ or $B \leq_{\mathsf{Lip}(d)} \neg A$, as desired.

Finally, let us show that the $\mathsf{Lip}(d)$-hierarchy on Borel subsets of X is also well-founded. Suppose not, and let $(A_n)_{n \in \omega}$ be a sequence of Borel subsets of X such that $A_{n+1} <_{\mathsf{Lip}(d)} A_n$ for every $n \in \omega$. Notice that this in particular implies that $A_n \neq \emptyset, X$ for every $n \in \omega$. By Claim 3.17.1 and our choice of the A_n's, for all $i < j$ there is no Lipschitz $f \colon (X', d_{X'}) \to (X', d_{X'})$ reducing $A_i \cap X'$ to $A_j \cap X'$. Using Ramsey's theorem, we get that there is an infinite $I \subseteq \omega$ such that either $\forall i, j \in I \, (i < j \Rightarrow A_j \cap X' \leq_{\mathsf{Lip}(d_{X'})} A_i \cap X')$, or else $\forall i, j \in I \, (i < j \Rightarrow A_j \cap X' \nleq_{\mathsf{Lip}(d_{X'})} A_i \cap X')$: in the former case the sequence $(A_i \cap X')_{i \in \omega}$ would give an infinite (strictly) descending chain in the $\mathsf{Lip}(d_{X'})$-hierarchy on X', while in the latter it would give an infinite antichain (in the same hierarchy). Since X' has bounded diameter and all the sets $A_i \cap X'$ are clearly Borel in it, both possibilities contradicts Theorem 3.10, and hence we are done. $\qquad\square$

Corollary 3.18. *Let $X = (X, d)$ be an ultrametric Polish space. Then the following are equivalent:*

    (1) *the diameter of X is nontrivially unbounded;*

    (2) *$(\mathscr{P}(\omega), \subseteq^*)$ embeds into $\mathsf{Deg}_{\mathbf{\Delta}_1^0}(\mathsf{Lip}(d))$;*

    (3) *the $\mathsf{Lip}(d)$-hierarchy on Borel (equivalently, clopen) subsets of X is very bad;*

    (4) *the $\mathsf{Lip}(d)$-hierarchy on Borel (equivalently, clopen) subsets of X is not very good.*

Hence $\mathsf{Deg}_{\mathbf{\Delta}_1^1}(\mathsf{Lip}(d))$ is either very good or very bad.

Proof. By Theorem 3.14 and Theorem 3.17. □

Corollary 3.19. *Let $X = (X, d)$ be a* perfect *ultrametric Polish space. Then*

    (1) *X has bounded diameter $\iff$ the $\mathsf{Lip}(d)$-hierarchy on Borel (equivalently, clopen) subsets of X is very good;*

    (2) *X has unbounded diameter $\iff$ the $\mathsf{Lip}(d)$-hierarchy on Borel (equivalently, clopen) subsets of X is very bad, and in fact in this case the partial order $(\mathscr{P}(\omega), \subseteq^*)$ embeds into $\mathsf{Deg}_{\mathbf{\Delta}_1^0}(\mathsf{Lip}(d))$.*

Let us consider again the ultrametrics d_ϕ introduced in Definition 3.2.

Corollary 3.20. *Let $\phi \colon \omega \to \mathbb{R}^+$ have unbounded range and suppose that $\inf \mathrm{rg}(\phi) > 0$. Then $(\mathscr{P}(\omega), \subseteq^*)$ embeds into both the $\mathsf{Lip}(d_\phi)$- and $\mathsf{L}(d_\phi)$-hierarchy on clopen subsets of $^\omega\omega$, and therefore both $\mathsf{Deg}_{\mathbf{\Delta}_1^0}(\mathsf{Lip}(d_\phi))$ and $\mathsf{Deg}_{\mathbf{\Delta}_1^0}(\mathsf{L}(d_\phi))$ are very bad. Conversely, if ϕ has bounded range, then the $\mathsf{Lip}(d_\phi)$-hierarchy $\mathsf{Deg}_{\mathbf{\Delta}_1^1}(\mathsf{Lip}(d_\phi))$ on Borel subsets of $^\omega\omega$ is very good.*

Proof. Observe that $(^\omega\omega, d_\phi)$ is a perfect ultrametric Polish space, and that it has unbounded diameter if and only if the $\mathrm{rg}(\phi)$ is unbounded in $\mathbb{R}^+$; then apply Theorems 3.14 and 3.10. □

4. Nonexpansive reducibilities

Definition 4.1. Let $X = (X, d)$ be an ultrametric Polish space. We say that $R(d)$ contains an *honest increasing sequence* if it contains a strictly increasing sequence $(r_n)_{n \in \omega}$ such that for some sequences $(x_n)_{n \in \omega}$, $(y_n)_{n \in \omega}$ of points in X the following conditions holds:

  (i) $d(x_n, x_m) = r_{\max\{n,m\}}$ for all distinct $n, m \in \omega$;

  (ii) $d(x_0, y_0) < r_0$ and $d(x_{n+1}, y_{n+1}) < d(x_n, y_n)$ for all $n \in \omega$.

The above condition is somewhat technical, but in case $X = (X, d)$ is a perfect ultrametric Polish space it is immediate to check that $R(d)$ contains an honest increasing sequence if and only if one of the following equivalent[11] conditions are satisfied:

    (1) there is $X' \subseteq X$ such that $R(d_{X'})$ has order type ω (with respect to the usual ordering on $\mathbb{R}$);

    (2) there is a sequence $(x_n)_{n \in \omega}$ of points in X and a strictly increasing sequence $(r_n)_{n \in \omega}$ of distances in $R(d)$ such that $d(x_n, x_m) = r_{\max\{n,m\}}$ for all distinct $n, m \in \omega$.

---

[11]To see that these two conditions are indeed equivalent, argue as in the first part of the proof of Theorem 3.14.

Notice also that if the diameter of an ultrametric Polish space $X = (X, d)$ is nontrivially unbounded, then $R(d)$ contains an honest increasing sequence by the first part of the proof of Theorem 3.14.

Theorem 4.2. *Let $X = (X, d)$ be a ultrametric Polish space such that $R(d)$ contains an honest increasing sequence. Then there is a map ψ from $\mathscr{P}(\omega)$ into the clopen subsets of X such that for all $a, b \subseteq \omega$*

$$a \subseteq^* b \iff \psi(a) \leq_{\mathsf{L}(d)} \psi(b).$$

Proof. Argue similarly to Theorem 3.14, with the following variations:

(a) let the sequences $(x_n)_{n \in \omega}$, $(y_n)_{n \in \omega}$, and $(r_n)_{n \in \omega}$ constructed at the beginning of the proof of Theorem 3.14 be witnesses of the fact that $R(d)$ contains an honest increasing sequence (forgetting about the extra properties required in Theorem 3.14), and set $s_n = d(x_n, y_n)$;[12]

(b) given $a \subseteq \omega$, define $\psi(a)$ as before, i.e. set $\psi(a) = \bigcup_{i \in \hat{a}} B_d(x_i, s_i)$, where $\hat{a} = \{2i \mid i \in \omega\} \cup \{2i + 1 \mid i \in a\}$;

(c) to prove the backward direction, use an argument similar to that of Theorem 3.14, but dropping any reference to the integer n (this simplification can be adopted here because we have to deal only with nonexpansive functions). More precisely: let f be a nonexpansive reduction of $\psi(a)$ to $\psi(b)$. Then for every $i \in \hat{a}$ there is a unique $j \in \omega$ such that $f(B_d(x_i, s_i)) \subseteq B_d(x_j, s_j)$ (because of the choice of the x_i, y_i's and the fact that f is nonexpansive). Arguing as in Claim 3.14.3, one immediately sees that we cannot have $j < i$ because in such case $s_i \leq s_j$. Conclude as in the final part of the proof of Theorem 3.14, using the fact that $r_k < r_j$ for every $j > k$. $\square$

Corollary 4.3. *There is an ultrametric Polish space $X_1 = (X_1, d_1)$ whose set of nonzero distances $R(d_1)$ is bounded away from 0 (hence it is countable and discrete) such that $(\mathscr{P}(\omega), \subseteq^*)$ embeds into the $\mathsf{L}(d_1)$-hierarchy on (clopen subsets of) X_1. Therefore $\mathsf{Deg}(\mathsf{L}(d_1)) = \mathsf{Deg}_{\mathbf{\Delta}_1^0}(\mathsf{L}(d_1))$ is very bad.*

Proof. Let $X_1 = \{x_n^i \mid n \in \omega, i = 0, 1\}$ and set

$$d_1(x_n^i, x_m^j) = \begin{cases} 0 & \text{if } n = m \text{ and } i = j \\ \frac{1}{2} + 2^{-(n+1)} & \text{if } n = m \text{ and } i \neq j \\ 2 - 2^{-\max\{n, m\}} & \text{if } n \neq m. \end{cases}$$

It is easy to check that $X_1 = (X_1, d_1)$ is an ultrametric Polish space. Moreover $r \geq \frac{1}{2}$ for every $r \in R(d_1)$, hence $R(d_1)$ is bounded away from 0. Moreover, the sequences obtained by setting $r_n = 2 - 2^{-n}$, $x_n = x_n^0$, and $y_n = x_n^1$ witness that $R(d_1)$ contains an honest increasing sequence. Hence the result follows from Theorem 4.2. $\square$

Remark 4.4. Notice that if an ultrametric Polish space $X = (X, d)$ satisfies the hypothesis of Corollary 4.3 (i.e. it is such that $R(d)$ is bounded away from 0), then its $\mathsf{Lip}(d)$-hierarchy is always (trivially) very good by Theorem 3.17 and the fact that all its points are ε-isolated for $\varepsilon = \inf R(d) > 0$.

---

[12]Clearly, the points x_n and y_n can again be chosen in any given countable dense set $Q \subseteq X$.

Corollary 4.5. *Given* $\phi\colon \omega \to \mathbb{R}^+$ *such that* $\inf \mathrm{rg}(\phi) > 0$, *if* $\mathrm{rg}(\phi)$ *contains an increasing* ω*-sequence then* $(\mathscr{P}(\omega), \subseteq^*)$ *embeds into the* $\mathsf{L}(d_\phi)$*-hierarchy on clopen subsets of* $^\omega\omega$, *and therefore* $\mathsf{Deg}_{\mathbf{\Delta}^0_1}(\mathsf{L}(d_\phi))$ *is very bad.*

Proof. Notice that $(^\omega\omega, d_\phi)$ is always a perfect Polish space, and that $R(d_\phi)$ has an honest increasing sequence if and only if $\mathrm{rg}(\phi)$ contains an increasing ω-sequence. Then apply Theorem 4.2. $\qquad\square$

Proposition 4.6. *Suppose that* $X = (X, d)$ *is an ultrametric Polish space such that* $R(d)$ *is either finite or a descending* (ω*-*)*sequence converging to* 0, *let* $I \leq \omega$ *be the cardinality of* $R(d)$, *and let* ρ *be the unique order-preserving map from* $\{2^{-i} \mid i < I\}$ *and* $R(d)$. *Then there is a closed set* $C \subseteq {}^\omega\omega$ *and a bijection* $f\colon C \to X$ *such that for all* $x, y \in X$

$$(*) \qquad\qquad d(x, y) = \rho(\bar{d}(f^{-1}(x), f^{-1}(y))).$$

In particular, the structures $(\mathscr{P}(X), \leq_{\mathsf{L}(d)}, \neg)$ *and* $(\mathscr{P}(C), \leq_{\mathsf{L}(\bar{d})}, \neg)$ *are isomorphic.*

Proof. Let us first assume that $I = \omega$, i.e. that $R(d)$ is a descending (ω-)sequence converging to 0. Inductively define the family $(A_s)_{s \in {}^{<\omega}\omega}$ of subsets of X by induction on $\mathrm{lh}(s)$ as follows. Set $A_\emptyset = X$. Then let $\{B_{s,j} \mid j < J\}$ (for some $J \leq \omega$) be an enumeration without repetitions of the collection $\{B_d(x, \rho(2^{-\mathrm{lh}(s)})) \mid x \in A_s\}$, and set $A_{s^\frown j} = B_{s,j}$ if $j < J$ and $A_{s^\frown j} = \emptyset$ otherwise. It is easy to check that the family $(A_s)_{s \in {}^{<\omega}\omega}$ is a Luzin scheme with vanishing diameter consisting of clopen sets. Hence letting $C = \{x \in {}^\omega\omega \mid \bigcap_{n \in \omega} A_{x \restriction n} \neq \emptyset\}$ and $f\colon C \to X$ be defined by letting $f(x)$ be the unique element of $\bigcap_{n \in \omega} A_{x \restriction n}$, we get that C and f are as required.

Assume now that I is finite, so that, in particular, X is a discrete space. Inductively define the sets A_s as above for all $s \in {}^{<\omega}\omega$ of length $\leq I$. Then if $\mathrm{lh}(s) = I$ the set A_s is either empty or a singleton. Letting $C = \{s^\frown 0^{(\omega)} \mid \mathrm{lh}(s) = I, A_s \neq \emptyset\}$ and defining $f\colon C \to X$ by letting $f(s^\frown 0^{(\omega)})$ be the unique element of A_s we again have that C and f are as required.

For the last part, notice that the map $\mathscr{P}(X) \to \mathscr{P}(C)\colon A \mapsto f^{-1}(A)$ is the desired isomorphism. To see this, simply notice that $(*)$ implies that $\mathsf{L}(d) = \{f \circ h \circ f^{-1} \mid h \in \mathsf{L}(\bar{d}_C)\}$. $\qquad\square$

Theorem 4.7. *Suppose that* $X = (X, d)$ *is an ultrametric Polish space such that* $R(d)$ *is either finite or a descending* (ω*-*)*sequence converging to* 0. *Then the* $\mathsf{L}(d)$*-hierarchy* $\mathsf{Deg}_{\mathbf{\Delta}^1_1}(\mathsf{L}(d))$ *on Borel subsets of* X *is very good.*

Proof. By Proposition 4.6, it is clearly enough to show that the $\mathsf{L}(\bar{d}_C)$-hierarchy on Borel subsets of C is very good: but this easily follows from the existence of a nonexpansive retraction of $(^\omega\omega, \bar{d})$ onto $(C, \bar{d}_C)$, Lemma 2.4, and the fact that the $\mathsf{L}(\bar{d})$-hierarchy on the Borel subsets of $^\omega\omega$ is very good. $\qquad\square$

Corollary 4.8. *Let* $\phi\colon \omega \to \mathbb{R}^+$ *and suppose that* $\mathrm{rg}(\phi)$ *is finite (so that trivially* $\inf \mathrm{rg}(\phi) > 0$). *Then the* $\mathsf{L}(d_\phi)$*-hierarchy* $\mathsf{Deg}_{\mathbf{\Delta}^1_1}(\mathsf{L}(d_\phi))$ *on Borel subsets of* $^\omega\omega$ *is very good.*

Proof. Simply observe that under our assumptions the set $R(d_\phi)$ is always an ω-sequence converging to 0, and then apply Theorem 4.7. $\qquad\square$

Let us now consider the general problem of determining the character of the $\mathsf{L}(d_\phi)$-hierarchy on Borel subsets of $^\omega\omega$ for an arbitrary $\phi\colon \omega \to \mathbb{R}^+$ with $\inf \mathrm{rg}(\phi) > 0$. By Corollary 4.5, if $\mathrm{rg}(\phi)$ contains an increasing ω-sequence, then $\mathsf{Deg}_{\boldsymbol{\Delta}^1_1}(\mathsf{L}(d_\phi))$ is very bad, hence we can assume without loss of generality that $\mathrm{rg}(\phi)$ has order type[13] α^* for some countable ordinal α. Corollary 4.8 considered the subcase where α is finite: the next proposition considers instead the special (but yet significant) subcase where $\alpha = \omega$ and ϕ is injective.

Notation 4.9. Given a set $A \subseteq {}^\omega\omega$ and a finite sequence $s \in {}^{<\omega}\omega$, let $s^\frown A = \{s^\frown x \mid x \in A\}$. When $\mathrm{lh}(s) = 1$, we simplify the notation by setting $n^\frown A = \langle n\rangle^\frown A$, and with a little abuse of notation we set $r^\frown A = \{r^\frown x \mid x \in A\} \subseteq \{r\} \times {}^\omega\omega$ also when r is not a natural number. Finally, given a family $(A_n)_{n\in\omega}$ of subsets of $^\omega\omega$, we set $\bigoplus_{n\in\omega} A_n = \bigcup_{n\in\omega} n^\frown A_n$.

Theorem 4.10. *Let $\phi\colon \omega \to \mathbb{R}^+$ be such that $\inf \mathrm{rg}(\phi) > 0$, and suppose that ϕ is injective and that $\mathrm{rg}(\phi)$ has order type ω^*. Then the $\mathsf{L}(d_\phi)$-hierarchy $\mathsf{Deg}_{\boldsymbol{\Delta}^1_1}(\mathsf{L}(d_\phi))$ on the Borel subsets of $^\omega\omega$ is very good.*

Proof. Using the usual game-theoretic arguments (see e.g. [And07]), it is easy to see that if a Borel $A \subseteq {}^\omega\omega$ is $\mathsf{L}(\bar{d})$-selfdual, then its $\mathsf{L}(\bar{d})$-degree $[A]_{\mathsf{L}(\bar{d})}$ is followed by an ω_1-chain of $\mathsf{L}(\bar{d})$-selfdual degrees $(\mathscr{L}^{(\alpha)}[A]_{\mathsf{L}(\bar{d})})_{\alpha<\omega_1}$, where the $\mathscr{L}^{(\alpha)}[A]_{\mathsf{L}(\bar{d})}$ are recursively defined as follows:

  (i) $\mathscr{L}^{(0)}[A]_{\mathsf{L}(\bar{d})} = [A]_{\mathsf{L}(\bar{d})}$;
  (ii) $\mathscr{L}^{(\alpha+1)}[A]_{\mathsf{L}(\bar{d})} = [0^\frown C]_{\mathsf{L}(\bar{d})}$ for some/any $C \in \mathscr{L}^{(\alpha)}[A]_{\mathsf{L}(\bar{d})}$;
  (iii) for limit α's, $\mathscr{L}^{(\alpha)}[A]_{\mathsf{L}(\bar{d})} = [\bigoplus_{n\in\omega} C_n]_{\mathsf{L}(\bar{d})}$, where $C_n \in \mathscr{L}^{(\alpha_n)}[A]_{\mathsf{L}(\bar{d})}$ for each $n \in \omega$, and $(\alpha_n)_{n\in\omega}$ is some/any increasing sequence cofinal in α.

We will use the following known facts about the Baire space $(^\omega\omega, \bar{d})$.

- A set A is *self-contractible* (i.e. reducible to itself via a contraction) if and only if it is $\mathsf{L}(\bar{d})$-nonselfdual; in this case the iterates of the contraction are reductions of A to itself and have a unique common fixed point (see [MR12, Corollary 4.4]).
- The $\mathsf{L}(\bar{d})$-nonselfdual degrees coincide with the $\mathsf{W}(\bar{d})$-nonselfdual degrees (see e.g. [VW78, Theorem 3.1]).
- Every $\mathsf{Lip}(\bar{d})$-selfdual degree $[A]_{\mathsf{Lip}(\bar{d})}$ is of the form $\bigcup\{[0^{(n)\frown}A']_{\mathsf{L}(\bar{d})} \mid n < \omega\}$ for some $\mathsf{L}(\bar{d})$-selfdual set A'; if instead $[A]_{\mathsf{Lip}(\bar{d})}$ is $\mathsf{Lip}(\bar{d})$-nonselfdual, then $[A]_{\mathsf{Lip}(\bar{d})} = [A]_{\mathsf{L}(\bar{d})}$ (see [MR10a]).
- If $A <_{\mathsf{Lip}(\bar{d})} B$, then for all $\varepsilon \in \mathbb{R}^+$ there is a Lipschitz reduction of A to B with constant ε (see the end of Section 4 in [MR12]).
- Let $\mathsf{W} = \mathsf{W}(^\omega\omega)$ be the set of all continuous functions from $^\omega\omega$ into itself, which is clearly a reducibility. Then every W-selfdual degree $[A]_\mathsf{W}$ is of the form $\bigcup\{\mathscr{L}^{(\alpha)}([A']_{\mathsf{L}(d)}) \mid \alpha < \omega_1\}$ for some $\mathsf{L}(\bar{d})$-selfdual set A' (see e.g. [And07]).

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[13]Given a linear order $L = (L, \leq)$, we denote by L^* the reverse linear oder induced by L, i.e. $L^* = (L, \leq^{-1})$. Since $\alpha = \{\beta \mid \beta < \alpha\}$ (for every ordinal α), we tacitly identify α with the linear order $\alpha = (\alpha, \leq)$, so that $\alpha^* = (\alpha, \geq)$.

Note that $({}^{\omega}\omega, d_{\phi})$ is isometric to the space $Y = \bigcup_{r \in \mathrm{rg}(\phi)} r^{\frown \omega}\omega$ equipped with the ultrametric (which with a little abuse of notation will be denoted by d_{ϕ} again)

$$d_{\phi}(r^{\frown}x, s^{\frown}y) = \begin{cases} 0 & \text{if } r = s \text{ and } x = y, \\ \max\{r, s\} & \text{if } r \neq s, \\ r \cdot 2^{-(n+1)} & \text{if } r = s \text{ and } n \text{ is least such that } x(n) \neq y(n). \end{cases}$$

Claim 4.10.1. *Every Borel subset $\bar{C}$ of $Y = (Y, d_{\phi})$ is $\mathsf{L}(d_{\phi})$-equivalent to one of the following ($\mathsf{L}(d_{\phi})$-)normal forms $\bar{A}$ (where in what follows $A, A_n \subseteq {}^{\omega}\omega$ and $<$ is the usual order on the reals):*

(1) $\bar{A} = \bigcup_{n \in \omega} r_n^{\frown} A_n$, where the sequence of the A_n's is $<_{\mathsf{Lip}(\bar{d})}$-increasing and the sequence $(r_n)_{n \in \omega}$ in $\mathrm{rg}(\phi)$ is strictly $<$-decreasing.

(2) $\bar{A} = \bigcup_{n \in \omega} r_n^{\frown} A_n$, where the sequence of the A_n's is $<_{\mathsf{L}(\bar{d})}$-increasing, $A_m \equiv_{\mathsf{Lip}(\bar{d})} A_n$ for all $m, n \in \omega$, and the sequence $(r_n)_{n \in \omega}$ in $\mathrm{rg}(\phi)$ is strictly $<$-decreasing.

(3) A is $\mathsf{L}(\bar{d})$-nonselfdual and
  (a) $\bar{A} = r^{\frown} A$ for some $r \in \mathrm{rg}(\phi)$, or
  (b) $\bar{A} = (r_0^{\frown} A) \cup (r_1^{\frown} (\neg A))$ for some $r_0, r_1 \in \mathrm{rg}(\phi)$ with $r_0 > r_1$, or
  (c) $\bar{A} = \bigcup_{i \in \omega} r_{2i}^{\frown} A \cup \bigcup_{i \in \omega} r_{2i+1}^{\frown}(\neg A)$ for some strictly $<$-decreasing sequence $(r_n)_{n \in \omega}$ in $\mathrm{rg}(\phi)$.

(4) A is $\mathsf{L}(\bar{d})$-selfdual and
  (a) $\bar{A} = r^{\frown} A$ for some $r \in \mathrm{rg}(\phi)$, or
  (b) $\bar{A} = \bigcup_{n \in \omega} r_n^{\frown} A$ for some strictly $<$-decreasing sequence $(r_n)_{n \in \omega}$ in $\mathrm{rg}(\phi)$.

Proof of the Claim. Let us sketch how to obtain these normal forms. We will often use the following easy fact. Let $D \subseteq \mathrm{rg}(\phi)$, $\rho\colon D \to \mathrm{rg}(\phi)$ be a non-$<$-increasing map, $\{f_r\colon {}^{\omega}\omega \to {}^{\omega}\omega \mid r \in D\} \subseteq \mathsf{L}(\bar{d})$, and $f'\colon \bigcup_{r \in \mathrm{rg}(\phi)\setminus D}(r^{\frown \omega}\omega) \to Y$ be a nonexpansive map (with respect to d_{ϕ}): then the map $f\colon Y \to Y$ defined by

$$f(r^{\frown} x) = \begin{cases} \rho(r)^{\frown} f_r(x) & \text{if } r \in D \\ f'(r^{\frown} x) & \text{otherwise} \end{cases}$$

is in $\mathsf{L}(d_{\phi})$.

Now let $\bar{C} = \bigcup_{r \in \mathrm{rg}(\phi)}(r^{\frown} C_r)$ be an arbitrary Borel subset of (Y, d_{ϕ}), and set $\mathcal{C} = \{C_r \mid r \in \mathrm{rg}(\phi)\}$, so that each C_r is a Borel subset of ${}^{\omega}\omega$. If $\mathcal{C}$ has no $\mathsf{Lip}(\bar{d})$-maximal element, choose a strictly $<$-decreasing sequence $(r_n)_{n \in \omega}$ in $\mathrm{rg}(\phi)$ such that $(C_{r_n})_{n \in \omega}$ is strictly $<_{\mathsf{Lip}(d_{\phi})}$-increasing and $<_{\mathsf{Lip}(d_{\phi})}$-cofinal in $\mathcal{C}$. Then $\bar{A} = \bigcup_{n \in \omega} C_{r_n}$ is in the normal form (1), and moreover it is easy to see that $\bar{A} \equiv_{\mathsf{L}(d_{\phi})} \bar{C}$. Otherwise, if $\mathcal{C}$ has a $\mathsf{Lip}(\bar{d})$-maximal element but no $\mathsf{L}(\bar{d})$-maximal element, then we can similarly find a set $\bar{A}$ in the normal form (2) which is $\mathsf{L}(d_{\phi})$-equivalent to $\bar{C}$.

Now suppose that there is an $\mathsf{L}(\bar{d})$-maximal element B among the sets in $\mathcal{C}$. Suppose first that B is $\mathsf{L}(\bar{d})$-nonselfdual.

If there is no $C \in \mathcal{C}$ with $C \equiv_{\mathsf{L}(\bar{d})} \neg B$, then we choose some $r \in \mathrm{rg}(\phi)$ with $C_r \equiv_{\mathsf{L}(\bar{d})} B$. Using the assumption $\inf \mathrm{rg}(\phi) > 0$ and the fact mentioned at the beginning of the proof that $\mathsf{L}(\bar{d})$-nonselfdual sets are self-contractible with arbitrarily small Lipschitz constant, it follows that $\bar{A} = r^{\frown} C_r \equiv_{\mathsf{L}(d_{\phi})} \bar{C}$, and $\bar{A}$ is in the normal form (3a). (For the nontrivial reduction, for each $t \in \mathrm{rg}(\phi)$ choose a $\mathsf{L}(\bar{d})$-reduction

$f_t\colon {}^\omega\omega \to {}^\omega\omega$ of C_t to C_r, let $\varepsilon \in \mathbb{R}^+$ be such that $\max \mathrm{rg}(\phi) \cdot \varepsilon \leq \inf \mathrm{rg}(\phi)$, and let $g\colon {}^\omega\omega \to {}^\omega\omega$ be a Lipschitz map with constant ε reducing C_r to itself. Define $f\colon Y \to Y$ by setting $f(t^\frown x) = r^\frown g(f_t(x))$ for every $t \in \mathrm{rg}(\phi)$ and $x \in {}^\omega\omega$: it is easy to check that $f \in \mathsf{L}(d_\phi)$ reduces $\bar{C}$ to $\bar{A}$.)

If there is a $<$-minimal $s \in \mathrm{rg}(\phi)$ with $C_s \equiv_{\mathsf{L}(\bar{d})} \neg B$, let r be either the $<$-minimal element of $\mathrm{rg}(\phi)$ with $C_r \equiv_{\mathsf{L}(\bar{d})} B$, or the $<$-largest element of $\mathrm{rg}(\phi)$ satisfying both $C_r \equiv_{\mathsf{L}(\bar{d})} B$ and $r < s$. Then $\bar{A} = r^\frown C_r \cup s^\frown C_s$ is in the normal form (3b), and arguing as above one can check that $\bar{C} \equiv_{\mathsf{L}_{d(\phi)}} \bar{A}$ using the assumption $\inf \mathrm{rg}(\phi) > 0$ and the previously mentioned fact about self-contractions. (For the nontrivial reduction, notice that we can assume without loss of generality that $r < s$ (otherwise we simply switch the role of C_r and C_s). Let $D = \{t \in \mathrm{rg}(\phi) \mid C_t \equiv_{\mathsf{L}(\bar{d})} \neg B\}$, so that $s = \min D$. For $t \in \mathrm{rg}(\phi)$, set $\rho(t) = s$ if $t \in D$ and $\rho(t) = r$ otherwise. Let f_t be a $\mathsf{L}(\bar{d})$-reduction of C_t to C_s if $t \in D$ and of C_t to C_r otherwise. Let ε and g be as above. Then the map $f\colon Y \to Y$ defined by $f(t^\frown x) = s^\frown f_t(x)$ if $t \in D$ and $f(t^\frown x) = r^\frown g(f_t(x))$ otherwise is an $\mathsf{L}(d_\phi)$-reduction of $\bar{C}$ to $\bar{A}$.)

If there are unboundedly many $s \in \mathrm{rg}(\phi)$ with $C_s \equiv_{\mathsf{L}(\bar{d})} \neg B$ and an $<$-minimal $r \in \mathrm{rg}(\phi)$ with $C_r \equiv_{\mathsf{L}(\bar{d})} B$, argue as in the previous paragraph switching the role of B and r with, respectively, $\neg B$ and s.

In the remaining case there are unboundedly many $r \in \mathrm{rg}(\phi)$ with $C_r \equiv_{\mathsf{L}(\bar{d})} B$ and unboundedly many $s \in \mathrm{rg}(\phi)$ with $C_s \equiv_{\mathsf{L}(\bar{d})} \neg B$. In this situation it is easy to see that $\bar{C}$ is $\mathsf{L}(d_\phi)$-equivalent to a set $\bar{A}$ in the normal form (3c).

Finally, suppose that $\mathcal{C}$ has a $\mathsf{L}(\bar{d})$-maximal element B and that B is $\mathsf{L}(\bar{d})$-selfdual. It follows from the remarks at the beginning of the proof that there is an $\mathsf{L}(\bar{d})$-nonselfdual set A with $B \in \mathscr{L}^{(\lambda+n)}[A \oplus (\neg A)]_{\mathsf{L}(\bar{d})}$ for some $n \in \omega$ and $\lambda = 0$ or λ a countable limit ordinal. Set $D = \{r \in \mathrm{rg}(\phi) \mid C_r \in \bigcup_{j\in\omega} \mathscr{L}^{(\lambda+j)}[A \oplus (\neg A)]_{\mathsf{L}(\bar{d})}\}$, and define the *index* of any $r \in D$ as $i(r) = r \cdot 2^{-(j+1)}$, where j is the unique natural number such that $C_r \in \mathscr{L}^{(\lambda+j)}[A \oplus (\neg A)]_{\mathsf{L}(\bar{d})}$. Then for any $r, s \in D$ for which $i(r) \leq i(s)$ there is an $\mathsf{L}(d_\phi)$-map f such that $f(s^\frown {}^\omega\omega) \subseteq r^\frown {}^\omega\omega$ and f reduces $s^\frown C_s$ to $r^\frown C_r$.

Suppose first that there is $j \leq n$ such that $C_{r_m} \in \mathscr{L}^{(\lambda+j)}[A \oplus (\neg A)]_{\mathsf{L}(\bar{d})}$ for some strictly $<$-descending sequence $(r_m)_{m\in\omega}$ of distances in $\mathrm{rg}(\phi)$, and let k be the largest of such j's. If $n = k$, then $\bar{A} = \bigcup_{m\in\omega} r_m^\frown C_{r_m}$ is in the normal form (4b) and $\bar{A} \equiv_{\mathsf{L}(d_\phi)} \bar{C}$. If $n > k$, let r be $<$-smallest in $\mathrm{rg}(\phi)$ such that $C_r \equiv_{\mathsf{L}(\bar{d})} B$. If $\inf \mathrm{rg}(\phi) < r \cdot 2^{n-k}$, then using the fact that C_r is reducible to each of the C_{r_m}'s with some Lipschitz function with constant 2^{n-k} we get that $\bar{C} \equiv_{\mathsf{L}(d_\phi)} \bar{C}'$, where $\bar{C}' = \bar{C} \setminus \left(\bigcup_{t \geq r} t^\frown {}^\omega\omega\right)$. Applying recursively this same procedure, after finitely many steps we will end up with a set $\bar{C}^* \equiv_{\mathsf{L}(d_\phi)} \bar{C}$ such that either the C_{r_m} are $\mathsf{L}(\bar{d})$-maximal in $\mathcal{C}^*$, or else there is an $<$-smallest r such that C_r is $\mathsf{L}(\bar{d})$-maximal in $\mathcal{C}^*$, $C_r \in \mathscr{L}^{(\lambda+n^*)}[A \oplus (\neg A)]_{\mathsf{L}(\bar{d})}$ for some $k < n^* \leq n$, and $r \cdot 2^{n^*-k} \leq \inf \mathrm{rg}(\phi)$. In the former case we again easily get that $\bar{A} = \bigcup_{m\in\omega} r_m^\frown C_{r_m}$ is in the normal form (4b) and $\bar{A} \equiv_{\mathsf{L}(d_\phi)} \bar{C}^* \equiv_{\mathsf{L}(d_\phi)} \bar{C}$. In the latter case, we get that $\bar{A} = r^\frown C_r$ is in normal form (4a) and $\bar{A} \equiv_{\mathsf{L}(d_\phi)} \bar{C}^* \equiv_{\mathsf{L}(d_\phi)} \bar{C}$. (To see that $\bar{C}^* \leq_{\mathsf{L}(d_\phi)} \bar{A}$, which is the only nontrivial reduction, notice that we may assume without loss of generality that all the C_{r_m}'s equal a fixed set $C \neq {}^\omega\omega$, that $C_r = 0^{(n^*-k)\frown} C$, and that for $t \notin \{r\} \cup \{r_m \mid m \in \omega\}$ either $C_t = \emptyset$ or $C_t = 0^{(i_t+1)\frown} C$ for some $i_t < n^* - k$. Fix $t \in \mathrm{rg}(\phi)$. If $t \geq r$ then let $f_t\colon {}^\omega\omega \to {}^\omega\omega$ be a $\mathsf{L}(\bar{d})$-reduction of C_t to C_r. If $t = r_m$

for some $m \in \omega$, define f_t by setting $f_t(x) = 0^{(n^*-k)\frown}x$ for all $x \in {}^\omega\omega$. Finally, if $t < r$ and $t \neq r_m$, then let f_t be a constant map with value $0^{(n^*-k)\frown}y$ for some fixed $y \notin C$ if $C_t = \emptyset$, and otherwise set $f_t(x) = 0^{(n^*-k-i_t-1)\frown}x$ for all $x \in {}^\omega\omega$. Then the map $f \colon Y \to Y$ defined by setting $f(t^\frown x) = r^\frown f_t(x)$ for all $t \in \mathrm{rg}(\phi)$ and $x \in {}^\omega\omega$ is a $\mathsf{L}(d_\phi)$-reduction of $\bar{C}$ to $\bar{A}$.)

Therefore we may assume without loss of generality that D is finite. Actually, applying the standard arguments used above it is not difficult to see that we may also assume that there are $m \in \omega$, a strictly $<$-decreasing sequence $r_0, \ldots, r_m \in \mathrm{rg}(\phi)$, and a strictly decreasing sequence $n_0, \ldots, n_m \in \omega$ such that:

- $C_{r_k} \in \mathscr{L}^{(\lambda+n_k)}[A \oplus (\neg A)]_{\mathsf{L}(\bar{d})}$ for all $k \leq m$;
- $i(r_k) < i(r_{k+1})$ for all $k < m$;
- $C_t = \emptyset$ for all $t \geq r_m$ which are not of the form r_k for some $k \leq m$;
- $C_t <_{\mathsf{Lip}(\bar{d})} C_{r_m}$ for all $t < r_m$.

Assume first that $\lambda > 0$. Then without loss of generality we may assume that $C_{r_k} = 0^{(n_k)\frown} \bigoplus_{l \in \omega}(0^{(l)\frown}C'_l)$ for all $k \leq m$, where the C'_l's are strictly $\mathsf{L}(\bar{d})$-increasing subsets of ${}^\omega\omega$ such that their $\mathsf{L}(\bar{d})$-degrees are cofinal below $\mathscr{L}^{(\lambda)}[A \oplus (\neg A)]_{\mathsf{L}(\bar{d})}$. Notice that in this case $i(r_k)$ measures the d_ϕ-distance between each pair of subsets of C_{r_k} of the form $0^{(n_k)\frown}l^\frown 0^{(l)\frown}C'_l$. Assume first that there is $l \in \omega$ such that $C_t \leq_{\mathsf{L}(\bar{d})} C'_l$ for all $t < r_m$. Then it is not hard to see that $\bar{A} = r_0^\frown C_{r_0}$ is in normal form (4a) and $\bar{A} \equiv_{\mathsf{L}(d_\phi)} \bar{C}$. (An $\mathsf{L}(d_\phi)$-reduction f of $\bar{C}$ to $\bar{A}$ may be defined on sets of the form $t^\frown{}^\omega\omega$ for $t < r_m$ by fixing $l' \geq l$ such that $2^{-l'} \leq \inf \mathrm{rg}(\phi)$ and an $\mathsf{L}(\bar{d})$-reduction f_t of C_t to $C'_{l'}$, and then setting $f(t^\frown x) = r_0^\frown 0^{(n_0)\frown}l'^\frown 0^{(l')\frown}f_t(x)$; for $t \geq r_m$, the map f may be defined on $t^\frown{}^\omega\omega$ in the obvious way using the property of the $i(r_k)$'s mentioned above.) Now assume instead that the family $\{C_t \mid t < r_m\}$ is $\mathsf{L}(\bar{d})$-cofinal below $\bigoplus_{l \in \omega} C'_l \equiv_{\mathsf{L}(\bar{d})} \bigoplus_{l \in \omega} 0^{(l)\frown}C'_l \in \mathscr{L}^{(\lambda)}[A \oplus (\neg A)]_{\mathsf{L}(\bar{d})}$. Then using arguments similar to the one already applied, one gets that if $i(r_0) \leq \inf \mathrm{rg}(\phi)$ then we can again set $\bar{A} = r_0^\frown C_0$, so that $\bar{A}$ is in normal form (4a), and prove that $\bar{A} \equiv_{\mathsf{L}(d_\phi)} \bar{C}$, while if $i(r_0) > \inf \mathrm{rg}(\phi)$ then we may choose a strictly decreasing sequence $(t_h)_{h \in \omega}$ so that $t_0 < \min\{r_m, i(r_0)\}$ and the C_{t_h}'s are $\leq_{\mathsf{L}(\bar{d})}$-increasing, all in the same $\mathsf{Lip}(\bar{d})$-degree, and cofinal below $\bigoplus_{l \in \omega} 0^{(l)\frown}C'_l$, and then prove that $\bar{A} = \bigcup_{h \in \omega} t_h^\frown C_{t_h}$ is in normal form (2) and $\mathsf{L}(d_\phi)$-equivalent to $\bar{C}$.

Finally, let $\lambda = 0$. In this case we may assume without loss of generality that $C_{r_k} = 0^{(n_k)\frown}(A \oplus \neg A)$ for all $k \leq m$, and $i(r_k)$ measures the distance between the copies of A and $\neg A$ in C_{r_k}. Let us first suppose that there are arbitrarily small $r, s > \inf \mathrm{rg}(\phi)$ with $C_r \equiv_{\mathsf{L}(d_\phi)} A$ and $C_s \equiv_{\mathsf{L}(d_\phi)} \neg A$. If $i(r_0) \leq \inf \mathrm{rg}(\phi)$, we let $\bar{A} = r_0^\frown C_{r_0}$; then $\bar{A}$ is in the normal form (4a) and arguing as above we get $\bar{A} \equiv_{\mathsf{L}(d_\phi)} \bar{C}$. If $i(r_0) > \inf \mathrm{rg}(\phi)$, we choose a strictly decreasing sequence $(t_h)_{h \in \omega}$ in $\mathrm{rg}(\phi)$ with $t_0 < \min\{r_m, i(r_0)\}$, $C_{t_{2p}} \equiv_{\mathsf{L}(\bar{d})} A$ and $C_{t_{2p+1}} \equiv_{\mathsf{L}(\bar{d})} \neg A$, and let $\bar{A} = \bigcup_{h \in \omega} t_h^\frown C_{t_h}$. Then $\bar{A}$ is in the normal form (3c) and, arguing as in the case $\lambda > 0$, we get $\bar{A} \equiv_{\mathsf{L}(d_\phi)} \bar{C}$. Next, let us suppose that there are no $r, s < i(r_0)$ in $\mathrm{rg}(\phi)$ with $C_r \equiv_{\mathsf{L}(\bar{d})} A$ and $C_s \equiv_{\mathsf{L}(\bar{d})} \neg A$. Let $\bar{A} = r_0^\frown C_{r_0}$. Then $\bar{A}$ is in the normal form (4a), and using the self-contractibility of A and $\inf \mathrm{rg}(\phi) > 0$ we again obtain $\bar{A} \equiv_{\mathsf{L}(d_\phi)} \bar{C}$. Finally, suppose that there are $r, s < i(r_0)$ in $\mathrm{rg}(\phi)$ with $C_r \equiv_{\mathsf{L}(\bar{d})} A$ and $C_s \equiv_{\mathsf{L}(\bar{d})} \neg A$ and that there is an $<$-minimal $r \in \mathrm{rg}(\phi)$ with $C_r \equiv_{\mathsf{L}(\bar{d})} A$ (the analogous situation in which there is a minimal $r \in \mathrm{rg}(\phi)$ with $C_r \equiv_{\mathsf{L}(\bar{d})} \neg A$ can be treated similarly). We consider the $<$-smallest $s \in \mathrm{rg}(\phi)$ with $C_s \equiv_{\mathsf{L}(\bar{d})} \neg A$

if this exists, and any $s \in \mathrm{rg}(\phi)$ with $s < r$ and $C_s \equiv_{\mathsf{L}(\bar{d})} \neg A$ otherwise. Then $\bar{A} = r^\frown A \cup s^\frown(\neg A)$ is in the normal form (3b) and $\bar{A} \equiv_{\mathsf{L}(d_\phi)} \bar{C}$. $\qquad\qquad\square$

By Claim 4.10.1, to show that $\mathsf{SLO}^{\mathsf{L}(d_\phi)}$ holds for Borel subsets of Y it is enough to show that for every pair of Borel sets $\bar{A}$ and $\bar{B}$ in $\mathsf{L}(d_\phi)$-normal form, either $\bar{A} \leq_{\mathsf{L}(d_\phi)} \bar{B}$ or $\bar{B} \leq_{\mathsf{L}(d_\phi)} \neg\bar{A}$: we are now going to sketch the proof of this fact, by considering all the possible combinations of normal forms.

If $\bar{A}$ is in case (1) of the normal form, then it is $\mathsf{L}(d_\phi)$-selfdual, and hence semi-linearity is equivalent to showing that $\bar{A} \leq_{\mathsf{L}(d_\phi)} \bar{B}$ or $\bar{B} \leq_{\mathsf{L}(d_\phi)} \bar{A}$. Let $A' = \oplus_{n \in \omega} A_n$, so that $[A']_{\mathsf{L}(\bar{d})} = \sup_{n \in \omega}[A_n]_{\mathsf{L}(\bar{d})}$. First assume that $\bar{B}$ is either in normal form (1) or (2), and let $B' = \bigoplus_{n \in \omega} B_n$. If $A' <_{\mathsf{L}(\bar{d})} B'$ (equivalently, $A' <_{\mathsf{Lip}(\bar{d})} B'$), then we get $\bar{A} \leq_{\mathsf{L}(d_\phi)} \bar{B}$, and similarly switching the role of A and B. If instead $A' \equiv_{\mathsf{L}(\bar{d})} B'$, then we get $\bar{A} \equiv_{\mathsf{L}(d_\phi)} \bar{B}$. Assume now that $\bar{B}$ is either in normal form (3) or (4). Then using B in place of B' in the argument above (and noticing that either $A' \leq_{\mathsf{Lip}(\bar{d})} B$ or else $B \leq_{\mathsf{L}(\bar{d})} A_n$ for all sufficiently large $n \in \omega$) we get again that $\bar{A}$ is $\mathsf{L}(d_\phi)$-comparable with $\bar{B}$, as required.

Let now $\bar{A}$ be in normal form (2). If $\bar{B}$ is in normal form (2) too, arguing as in the previous case we compare $A' = \bigoplus_{n \in \omega} A_n$ and $B' = \bigoplus_{n \in \omega} B_n$ with respect to $\mathsf{L}(\bar{d})$. Similarly, if $\bar{B}$ is in case (3), we compare A' with B with respect to $\mathsf{L}(\bar{d})$, and then argue as above again. Now let us suppose that $\bar{B}$ is in case (4). If $A_n <_{\mathsf{Lip}(\bar{d})} B$ for all $n \in \omega$, then $\bar{A} \leq_{\mathsf{L}(d_\phi)} \bar{B}$. Otherwise $B \leq_{\mathsf{L}(\bar{d})} A_n$ for some $n \in \omega$ and thus $\bar{B} \leq_{\mathsf{L}(\bar{d})} \bar{A}$.

We now assume that $\bar{A}$ is in normal form (3). If $\bar{B}$ is in normal form (3) too, we can prove $\bar{A} \leq_{\mathsf{L}(d_\phi)} \bar{B}$ or $\bar{B} \leq_{\mathsf{L}(d_\phi)} \neg\bar{A}$ by first comparing A and B with respect to $\mathsf{Lip}(\bar{d})$-reducibility (equivalently, $\mathsf{L}(\bar{d})$-reducibility), using the assumption $\inf \mathrm{rg}(\phi) > 0$. If $\bar{A} \equiv_{\mathsf{Lip}(\bar{d})} \bar{B}$ and both $\bar{A}$ and $\bar{B}$ are in case (3b), then we simply compare the minimum of the values r_1 appearing in their normal forms. The comparison is straightforward in all other cases for $\bar{A}$ and $\bar{B}$ in the normal form (3) with $\bar{A} \equiv_{\mathsf{Lip}(\bar{d})} \bar{B}$. If instead $\bar{B}$ is in case (4), using the assumption $\inf \mathrm{rg}(\phi) > 0$, we simply need to compare the $\mathsf{L}(\bar{d})$-degrees of A and B; all possible relationships between these degrees with respect to $\leq_{\mathsf{L}(\bar{d})}$ can be transferred back to analogous relationships between $\bar{A}$ and $\bar{B}$ with respect to $\leq_{\mathsf{L}(d_\phi)}$.

Let us finally suppose that $\bar{A}$ and $\bar{B}$ are both in case (4); this is the more delicate case. Since $\bar{A}$ and $\bar{B}$ are clearly $\mathsf{L}(d_\phi)$-selfdual, it is again sufficient to show that they are $\mathsf{L}(d_\phi)$-comparable. First assume that $\bar{A}$ and $\bar{B}$ are both in case (4a), where $\bar{A} = r^\frown A$ and $\bar{B} = s^\frown B$. If A and B are not in the same $\mathsf{Lip}(\bar{d})$-degree, then it is easy to compare $\bar{A}$ and $\bar{B}$ with respect to $\leq_{\mathsf{L}(d_\phi)}$, and if $A \leq_{\mathsf{L}(\bar{d})} B$ then $\bar{A} \leq_{\mathsf{L}(d_\phi)} \bar{B}$. Hence we can assume that $A \equiv_{\mathsf{Lip}(\bar{d})} B$ and $B \leq_{\mathsf{L}(\bar{d})} A$, so that $A \equiv_{\mathsf{L}(\bar{d})} 0^{(n)\frown}B$ for some $n \in \omega$. Using the assumption $\inf \mathrm{rg}(\phi) > 0$, it is now easy to check that $\bar{A} \leq_{\mathsf{L}(d_\phi)} \bar{B}$ holds if $s \cdot 2^n \leq r$, while $\bar{B} \leq_{\mathsf{L}(d_\phi)} \bar{A}$ holds if $r \leq s \cdot 2^n$.

Suppose now that $\bar{A} = r^\frown A$ is in case (4a) and $\bar{B}$ is in case (4b). We have $\bar{A} \leq_{\mathsf{L}(d_\phi)} \bar{B}$ if $A \leq_{\mathsf{L}(\bar{d})} B$ holds, and moreover $B <_{\mathsf{Lip}(\bar{d})} A$ implies that $0^{(n)\frown}B \leq_{\mathsf{L}(\bar{d})} A$ for all $n \in \omega$ (which in turn implies $\bar{B} \leq_{\mathsf{L}(d_\phi)} \bar{A}$). Thus we can assume that $A \equiv_{\mathsf{Lip}(\bar{d})} B$ and $B \leq_{\mathsf{L}(\bar{d})} A$, so that again $A \equiv_{\mathsf{L}(\bar{d})} 0^{(n)\frown}B$ for some $n \in \omega$, and let $s = \inf \mathrm{rg}(\phi)$. Arguing similarly to the previous case, it is easy to check that $\bar{A} \leq_{\mathsf{L}(d_\phi)} \bar{B}$ holds if $s \cdot 2^n < r$, while $\bar{B} \leq_{\mathsf{L}(d_\phi)} \bar{A}$ holds if $s \cdot 2^n \geq r$.

The last case that needs to be considered is when both $\bar{A}$ and $\bar{B}$ are in case (4b). We may assume that $A \leq_{\mathsf{L}(\bar{d})} B$ and hence $\bar{A} \leq_{\mathsf{L}(d_\phi)} \bar{B}$. This concludes the proof that $\mathsf{SLO}^{\mathsf{L}(d_\phi)}$ holds for Borel subsets (in normal form) of Y.

It remains to show that the $\mathsf{L}(d_\phi)$-hierarchy on Borel subsets of Y is well-founded, and for this we may again concentrate only on sets in normal form. Assume towards a contradiction that there is a family $(\bar{A}^{(i)})_{i \in \omega}$ of Borel subsets of Y in normal form such that $\bar{A}^{(i+1)} <_{\mathsf{L}(d_\phi)} \bar{A}^{(i)}$ for all $i \in \omega$. Since there are only finitely many types of normal form, passing to a subsequence if necessary we may further assume that all the $\bar{A}^{(i)}$'s share the same type of normal form. We now consider the various possibilities.

First assume that the $\bar{A}^{(i)}$'s are all in normal form (1), and set $(A')^{(i)} = \bigoplus_{n \in \omega} A_n^{(i)}$, where the sets $A_n^{(i)} \subseteq {}^\omega\omega$ are those appearing in the normal form of $\bar{A}^{(i)}$. Notice that all the $(A')^{(i)}$ are necessarily $\mathsf{L}(\bar{d})$-selfdual. Then $(A')^{(i+1)} <_{\mathsf{L}(\bar{d})} (A')^{(i)}$, because otherwise $(A')^{(i)} \leq_{\mathsf{L}(\bar{d})} (A')^{(i+1)}$, whence one would easily get $\bar{A}^{(i)} \leq_{\mathsf{L}(d_\phi)} \bar{A}^{(i+1)}$, contradicting the choice of the $\bar{A}^{(i)}$'s. Therefore the $(A')^{(i)}$ are strictly $\mathsf{L}(\bar{d})$-decreasing, contradicting the fact that the $\mathsf{L}(\bar{d})$-hierarchy on Borel subsets of ${}^\omega\omega$ is well-founded.

The case where all the $\bar{A}^{(i)}$'s are in normal form (2) can be dealt with in the same way, and a similar argument works also for the other cases with the following minor modifications:

- When considering normal forms as in (3a), set $(A')^{(i)} = A^{(i)}$, where $A^{(i)} \subseteq {}^\omega\omega$ is the set appearing in the normal form of $\bar{A}^{(i)}$, and pass to a subsequence if necessary to avoid the situations in which $A^{(i+1)} \equiv_{\mathsf{L}(\bar{d})} \neg A^{(i)}$;

- When considering normal forms as in (3b) or (3c), set $(A')^{(i)} = (0^\frown A^{(i)}) \cup (1^\frown(\neg A^{(i)}))$, where $A^{(i)}, \neg A^{(i)} \subseteq {}^\omega\omega$ are the sets appearing in the normal form of $\bar{A}^{(i)}$. In case (3b), we may need to pass to a subsequence $((A')^{(i_l)})_{l \in \omega}$ to guarantee that $A^{(i_{l+1})} <_{\mathsf{L}(\bar{d})} A^{(i_l)}$.

- When considering normal forms as in (4), set $(A')^{(i)} = A^{(i)}$, where $A^{(i)} \subseteq {}^\omega\omega$ is the set appearing in the normal form of $\bar{A}^{(i)}$. In case (4a) it may be necessary to first pass to a subsequence $((A')^{(i_l)})_{l \in \omega}$ to guarantee that the sequence of the $r^{(i_l)}$'s appearing in the canonical form of $\bar{A}^{(i_l)}$ is not $<$-increasing.

This concludes the proof of the well-foundness of $\leq_{\mathsf{L}(d_\phi)}$ on Borel subsets of Y, and hence of the entire proposition. $\qquad\square$

Corollaries 4.5, 4.8, and Theorem 4.10 already cover many interesting cases, and using the methods developed in the proof of Theorem 4.10 it seems plausible to conjecture that if the range of ϕ does not contain increasing ω-sequences, then the $\mathsf{L}(d_\phi)$-hierarchy on Borel subsets of ${}^\omega\omega$ is well-founded. However, the general problem of determining the character of the $\mathsf{L}(d)$-hierarchy on an arbitrary ultrametric Polish space $X = (X, d)$ remains open:

Question 4.11. *Let $X = (X, d)$ be an ultrametric (perfect) Polish space such that $R(d)$ does not contain an honest increasing sequence, and assume that $R(d)$ is neither finite nor a (ω-)sequence converging to 0. Is the $\mathsf{L}(d)$-hierarchy $\mathsf{Deg}_{\mathbf{\Delta}_1^1}(\mathsf{L}(d))$ on the Borel subsets of X (very) good?*

Remark 4.12. In order to answer Question 4.11, it may be useful to note the following. It is proved in [GS11, Theorem 4.1] that every ultrametric Polish space

$X = (X, d)$ is isometric to a closed subspace of the ultrametric Urysohn space $U_{R(d)} = \{(x_n)_{n \in \omega} \in {}^\omega(R(d) \cup \{0\}) \mid x_n \geq x_{n+1}$ for all n and $\lim_{n \to \infty} x_n = 0\}$ equipped with the complete ultrametric

$$d_{U_{R(d)}}((x_n)_{n \in \omega}, (y_n)_{n \in \omega}) = \begin{cases} 0 & \text{if } x_n = y_n \text{ for all } n, \\ \max(x_n, y_n) & \text{if } n \text{ is least such that } x(n) \neq y(n). \end{cases}$$

Suppose that $X = (X, d)$ is a perfect ultrametric Polish space and choose a closed subspace Y of $(U_{R(d)}, d_{U_{R(d)}})$ such that $Y = (Y, d_{U_{R(d)}})$ is isometric to X. Let $S(Y) = \{y \restriction n \mid y \in Y, \ n \in \omega\}$, and set $D(s) = \{r \in \mathbb{R}^+ \mid \exists x \in {}^\omega(R(d) \cup \{0\}) \, (s^\frown r^\frown x \in Y)\}$ for each $s \in S(Y)$. Notice that $S(Y)$ and $D(s)$ are countable since $R(d)$ is countable. If there is a strictly increasing sequence $(r_n)_{n \in \omega}$ in $D(s)$ for some $s \in S(Y)$, then we obtain an honest increasing sequence in $R(d)$ from the assumption that (X, d) is perfect. If there is no honest increasing sequence in $R(d)$, it follows that the order type of each $D(s)$ is α_s^* for some countable ordinal α_s.

Finally, we want to show that, even if by Theorem 4.2 it is possible that the $\mathsf{L}(d)$-hierarchy $\mathsf{Deg}_{\mathbf{\Delta}_1^1}(\mathsf{L}(d))$ on Borel subsets of a given ultrametric Polish space $X = (X, d)$ with bounded diameter is very bad, a natural (modest) strengthening of the preorder $\leq_{\mathsf{L}(d)}$ already yields to a semi-linearly ordered hierarchy.

Definition 4.13. Suppose $X = (X, d)$ is an ultrametric Polish space, and let $A, B \subseteq X$. Let us write $A \leq_{\mathsf{Lip}(d, L)} B$ if there is a Lipschitz function $f \colon (X, d) \to (X, d)$ with constant $L \in \mathbb{R}^+$ such that $A = f^{-1}(B)$. We say that A is *almost nonexpansive reducible* to B ($A \leq_{\mathsf{aL}(d)} B$ in symbols) if $A \leq_{\mathsf{Lip}(d, L)} B$ for every $1 < L \in \mathbb{R}^+$.

Notice that the relation $\leq_{\mathsf{aL}(d)}$ is a preorder (for the transitivity use the fact that if $f, g \colon X \to X$ are Lipschitz functions with constant L, L', respectively, then $g \circ f$ is Lipschitz with constant $L \cdot L'$). Moreover, $\leq_{\mathsf{aL}(d)}$ is strictly between $\leq_{\mathsf{L}(d)}$ and $\leq_{\mathsf{Lip}(d)}$. Even if literally $\leq_{\mathsf{aL}(d)}$ is not of the form $\leq_{\mathcal{F}}$ for some reducibility $\mathcal{F}$ on X, with a little abuse of notation and terminology we can nevertheless consider the $\mathsf{aL}(d)$-hierarchy on (Borel subsets of) X, the Semi-Linear Ordering principle $\mathsf{SLO}^{\mathsf{aL}(d)}$, and so on (with the obvious definitions).

Proposition 4.14. *Let $X = (X, d)$ be an ultrametric Polish space with bounded diameter. Then the $\mathsf{aL}(d)$-hierarchy on the Borel subsets of X is semi-linearly ordered, and hence not bad.*

Proof. Given $L > 1$, let $d_L \colon X \times X \to \mathbb{R}^+$ be defined by $d_L(x, y) = \min(\{L^n \mid d(x, y) \leq L^n$ and $n \in \mathbb{Z}\})$ if $x, y \in X$ are distinct, and by $d_L(x, y) = 0$ if $x = y \in X$. Then d_L is a complete ultrametric on X compatible with the metric topology τ_d, and since we assumed that X has bounded (d-)diameter we also have that $R(d_L) \subseteq \{L^n \mid n \in \mathbb{Z}\}$ is either finite, or a decreasing sequence converging to 0. By Theorem 4.7, this means that the $\mathsf{L}(d_L)$-hierarchy on Borel subsets of X is very good, and hence, in particular, semi-linearly ordered. Moreover, $\mathrm{id} \colon (X, d) \to (X, d_L)$ is Lipschitz with constant L, while $\mathrm{id} \colon (X, d_L) \to (X, d)$ is nonexpansive. Hence for all subsets A, B of X:

- if $A \leq_{\mathsf{Lip}(d, L')} B$, then $A \leq_{\mathsf{Lip}(d_L, L \cdot L')} B$;
- if $A \leq_{\mathsf{Lip}(d_L, L')} B$, then $A \leq_{\mathsf{Lip}(d, L \cdot L')} B$.

In particular, $A \leq_{\mathsf{L}(d_L)} B$ implies that $A \leq_{\mathsf{Lip}(d, L)} B$.

We claim $\mathsf{SLO}^{\mathsf{aL}(d)}$ holds for Borel subsets of X. By the observation above and $\mathsf{SLO}^{\mathsf{Lip}(d_L)}$, for every fixed $L > 1$ we have that either $A \leq_{\mathsf{Lip}(d,L)} B$ or $B \leq_{\mathsf{Lip}(d,L)} \neg A$. If for every $n \in \omega$ there is $1 < L \leq 1 + 2^{-n}$ such that $A \leq_{\mathsf{Lip}(d,L)} B$, then $A \leq_{\mathsf{aL}(d)} B$. Similarly, if for every $n \in \omega$ there is $1 < L \leq 1 + 2^{-n}$ such that $B \leq_{\mathsf{Lip}(d,L)} \neg A$, then $B \leq_{\mathsf{aL}(d)} \neg A$. Since one of the two possibilities necessarily occurs, we get that either $A \leq_{\mathsf{aL}(d)} B$ or $B \leq_{\mathsf{aL}(d)} \neg A$, as required. $\qquad\square$

5. Compact ultrametric Polish spaces

It is well-known that any continuous function between metric spaces is automatically uniformly continuous as soon as its domain is compact (see e.g. [Kec95, Proposition 4.5]). In particular, this means that it does not make much sense to consider the $\mathsf{UCont}(d)$-hierarchy on a compact ultrametric Polish space $X = (X, d)$: since it coincide[14] with the $\mathsf{W}(X)$-hierarchy on X, its restriction to the Borel sets is always very good by Proposition 1.2. However, one may wonder about the character of the $\mathsf{Lip}(d)$- and the $\mathsf{L}(d)$-hierarchy on (Borel subsets of) such an X: the next results show that they must always be very good as well.

Proposition 5.1. *Let $X = (X, d)$ be a compact ultrametric Polish space. Then either X (and hence also $R(d)$) is finite, or else $R(d)$ is a strictly decreasing (ω-)sequence converging to 0. In particular, X has bounded diameter.*

Proof. It is clearly enough to show that for every $\bar{r} \in \mathbb{R}^+$, the set $R(d)_{\geq \bar{r}} = \{r \in R(d) \mid r \geq \bar{r}\}$ is finite. To see this, observe that the family $\mathcal{B} = \{B_d(x, \bar{r}) \mid x \in X\}$ is a finite covering of X because X is compact. Assume towards a contradiction that $R(d)_{\geq \bar{r}}$ is infinite, let $(r_n)_{n \in \omega}$ be an enumeration without repetitions of it, and let $(x_n)_{n \in \omega}$ and $(y_n)_{n \in \omega}$ be such that $d(x_n, y_n) = r_n$ for every $n \in \omega$. Since $\mathcal{B}$ is finite, there are distinct $n, m \in \omega$ such that $d(x_n, x_m), d(y_n, y_m) < \bar{r}$. Since $r_m \geq \bar{r}$, we get that $r_n = d(x_n, y_n) = d(x_m, y_m) = r_m$, contradicting the choice of the r_n's. $\qquad\square$

Theorem 5.2. *Let $X = (X, d)$ be a compact ultrametric Polish space. Then both the $\mathsf{L}(d)$- and the $\mathsf{Lip}(d)$-hierarchy on Borel subsets of X are very good.*

Proof. Use Proposition 5.1 together with Theorems 4.7 and 3.10. $\qquad\square$

In particular, we cannot change the ultrametric on the Cantor space ${}^\omega 2$ to make its nonexpansive or its Lipschitz hierarchy (very) bad: if d' is any complete ultrametric compatible with the product topology on ${}^\omega 2$, then both the $\mathsf{Lip}(d')$- and the $\mathsf{L}(d')$-hierarchy on Borel subsets of ${}^\omega 2$ are very good.[15]

Remark 5.3. Albeit Theorem 5.2 shows that there is no compact ultrametric Polish space $X = (X, d)$ with a (very) bad $\mathsf{Lip}(d)$- or $\mathsf{L}(d)$-hierarchy, Corollaries 3.16 and 4.3 shows that there are $\boldsymbol{K}_\sigma$-spaces[16] $X_i = (X_i, d_i)$, $i = 0, 1$, such that:

- both $\mathsf{Deg}_{\boldsymbol{\Delta}^0_1}(\mathsf{L}(d_0))$ and $\mathsf{Deg}_{\boldsymbol{\Delta}^0_1}(\mathsf{Lip}(d_0))$ are very bad;
- $\mathsf{Deg}_{\boldsymbol{\Delta}^0_1}(\mathsf{L}(d_1))$ is very bad, while $\mathsf{Deg}_{\boldsymbol{\Delta}^1_1}(\mathsf{Lip}(d_1))$ is very good.

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[14]In fact in the specific case of the Cantor space $\mathcal{C} = ({}^\omega 2, \bar{d}_\mathcal{C})$ one can check that, although $\mathsf{Lip}(\bar{d}_\mathcal{C}) \subsetneq \mathsf{UCont}(\bar{d}_\mathcal{C})$, the $\mathsf{Lip}(\bar{d}_\mathcal{C})$- and the $\mathsf{UCont}(\bar{d}_\mathcal{C})$-hierarchies coincide.

[15]However, analogously to [MR12, Section 5] it is still possible to define compatible complete ultrametrics d_i, $i = 0, 1$, on $\mathcal{C} = {}^\omega 2$ so that $\mathsf{L}(\bar{d}_\mathcal{C}) \not\subseteq \mathsf{Lip}(d_0)$ (hence also $\mathsf{L}(\bar{d}_\mathcal{C}) \not\subseteq \mathsf{L}(d_0)$), while $\mathsf{L}(\bar{d}_\mathcal{C}) \not\subseteq \mathsf{L}(d_1)$ but $\mathsf{Lip}(\bar{d}_\mathcal{C}) = \mathsf{Lip}(d_1)$.

[16]A topological space is $\boldsymbol{K}_\sigma$ if it is the union of countably many compact subsets.

Let us now concentrate on the Cantor space $\mathcal{C} = {}^\omega 2$, and let us briefly consider another kind of reducibility that was analyzed in [MR12] for the case of the Baire space, namely the collection of all contraction mappings.

Notation 5.4. Let $\bar{d} = \bar{d}_{\mathcal{C}}$ be the usual metric on the Cantor space. We denote by $\mathsf{c}(\bar{d})$ the collection of all *contractions* from $\mathcal{C}$ into itself, i.e. of all Lipschitz functions $f \colon \mathcal{C} \to \mathcal{C}$ with constant strictly smaller than 1.

Given two sets $A, B \subseteq \mathcal{C}$, set

$$A \leq_{\mathsf{c}(\bar{d})} B \iff A = B \vee \exists f \in \mathsf{c}(\bar{d})\,(A = f^{-1}(B)).$$

In fact, $\leq_{\mathsf{c}(\bar{d})} = \leq_{\mathcal{F}}$, where $\mathcal{F}$ is the reducibility on $\mathcal{C}$ obtained by adding the identity $\mathrm{id} = \mathrm{id}_{\mathcal{C}}$ to the set $\mathsf{c}(\bar{d})$.

Using the methods developed in [MR12, Section 4], it is easy to check that the following hold:

Theorem 5.5. *Let A, B be Borel subsets of $\mathcal{C}$.*

(1) *If $A \not\equiv_{\mathsf{L}(\bar{d})} B$, then $A \leq_{\mathsf{c}(\bar{d})} B \iff A \leq_{\mathsf{L}(\bar{d})} B$, while if $A \equiv_{\mathsf{L}(\bar{d})} B$, then $A \leq_{\mathsf{c}(\bar{d})} B \iff A \not\leq_{\mathsf{L}(\bar{d})} \neg A$.*

(2) *A is* selfcontractible *(i.e. $A = f^{-1}(A)$ for some $f \in \mathsf{c}(\bar{d})$) if and only if $A \not\leq_{\mathsf{L}(\bar{d})} \neg A$.*

(3) *If $A \not\leq_{\mathsf{L}(\bar{d})} \neg A$, then $[A]_{\mathsf{c}(\bar{d})} = [A]_{\mathsf{L}(\bar{d})}$, while if $A \leq_{\mathsf{L}(\bar{d})} \neg A$, then $[A]_{\mathsf{c}(\bar{d})} = \{A\}$.*

(4) *$A <_{\mathsf{c}(\bar{d})} B \iff A <_{\mathsf{L}(\bar{d})} B$.*

Therefore, to describe the $\mathsf{c}(\bar{d})$-hierarchy on Borel subsets of $\mathcal{C}$ it is enough to determine how many sets are contained in each $\mathsf{L}(\bar{d})$-degree of an $\mathsf{L}(\bar{d})$-selfdual Borel subset of $\mathcal{C}$, and to combine this information with the well-known description of the $\mathsf{L}(\bar{d})$-hierarchy on Borel subsets of $\mathcal{C}$ (see [And07]). Let us first briefly describe this last hierarchy. First of all, the hierarchy is semi-well-ordered. At the bottom we found the $\mathsf{L}(\bar{d})$-nonselfdual pair constituted by $[\mathcal{C}]_{\mathsf{L}(\bar{d})} = \{\mathcal{C}\}$ and $[\emptyset]_{\mathsf{L}(\bar{d})} = \{\emptyset\}$. Immediately after each $\mathsf{L}(\bar{d})$-nonselfdual pair $\{[A]_{\mathsf{L}(\bar{d})}, [\neg A]_{\mathsf{L}(\bar{d})}\}$ there is the $\mathsf{L}(\bar{d})$-degree of the $\mathsf{L}(\bar{d})$-selfdual set $A \oplus \neg A = (0^\frown A) \cup (1^\frown(\neg A)) = \{0^\frown x \mid x \in A\} \cup \{1^\frown x \mid x \in \neg A\}$. On the other hand, if A is $\mathsf{L}(\bar{d})$-selfdual, then immediately after $[A]_{\mathsf{L}(\bar{d})}$ there is the $\mathsf{L}(\bar{d})$-degree of the selfdual set $0^\frown A = \{0^\frown x \mid x \in A\}$. Finally, at all limit level there is always an $\mathsf{L}(\bar{d})$-nonselfdual pair. Therefore we get the structure represented in Figure 1, where bullets represent $\mathsf{L}(\bar{d})$-degrees and each $\mathsf{L}(\bar{d})$-degree is $\mathsf{L}(\bar{d})$-reducible to another one if and only if it is (strictly) to the left of it.
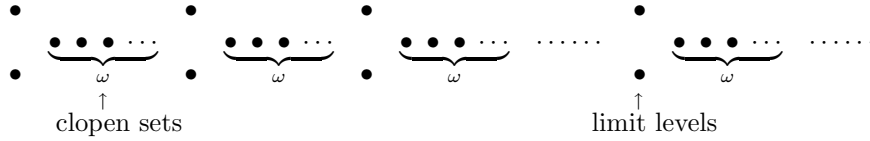

FIGURE 1. The $\mathsf{L}(\bar{d})$-hierarchy on Borel subsets of $\mathcal{C}$.

Notice that the first ω-chain of consecutive $\mathsf{L}(\bar{d})$-selfdual degrees contains all nontrivial clopen sets, while the first non-trivial $\mathsf{L}(\bar{d})$-nonselfdual pair is formed by all proper open and proper closed subsets of $\mathcal{C}$.

To compute the cardinality of a given $[A]_{\mathsf{L}(\bar{d})}$ (for $A \subseteq \mathcal{C}$), recall first that if $\emptyset, \mathcal{C} \neq A$ is clopen, then there is $0 \neq n \in \omega$, called the *level of A* such that $A \equiv_{\mathsf{L}(\bar{d})} \mathbf{N}_{0^{(n)}}$, where for an arbitrary $s \in {}^{<\omega}2$ we set $\mathbf{N}_s = \{x \in \mathcal{C} \mid s \subset x\}$ — in fact A is in the n-th $\mathsf{L}(\bar{d})$-selfdual degree of the first ω-chain of consecutive $\mathsf{L}(\bar{d})$-selfdual degrees if and only if it is of level n.

Proposition 5.6. *Let $\emptyset, \mathcal{C} \neq A \subseteq \mathcal{C}$.*

(1) *if A is clopen, then $[A]_{\mathsf{L}(\bar{d})}$ contains exactly $2^{2^n} - 2^{2^{n-1}}$-many sets, where n is the level of A;*
(2) *if A is not clopen, then there is an injection $j \colon \mathcal{C} \to [A]_{\mathsf{L}(\bar{d})}$.*

Proof. For each $0 \neq n \in \omega$, the collection of all clopen sets $\mathsf{L}(\bar{d})$-reducible to $\mathbf{N}_{0^{(n)}}$ consists of all the sets of the form $\bigcup_{s \in S} \mathbf{N}_s$ for S a subset of $\{s \in {}^{<\omega}2 \mid \mathrm{lh}(s) = n\}$: therefore there are 2^{2^n}-many such sets. So if A is a clopen set of level n, then to compute the cardinality of $[A]_{\mathsf{L}(\bar{d})}$ we have to subtract to 2^{2^n} the number of sets which are not $\mathsf{L}(\bar{d})$-equivalent to $\mathbf{N}_{0^{(n)}}$, i.e. $\emptyset, \mathcal{C}$, and all sets $\mathsf{L}(\bar{d})$-reducible to $\mathbf{N}_{0^{(n-1)}}$: since there are $2^{2^{n-1}}$-many such sets, we get that $[A]_{\mathsf{L}(\bar{d})}$ contains exactly $2^{2^n} - 2^{2^{n-1}}$-many sets.

For the second part, let us first assume that $A \not\leq_{\mathsf{L}(\bar{d})} \neg A$. If A is a proper open set, then the map $j \colon \mathcal{C} \to [A]_{\mathsf{L}(\bar{d})} \colon x \mapsto \mathcal{C} \setminus \{x\}$ is as required. Therefore we can assume without loss of generality that $B \leq_{\mathsf{L}(\bar{d})} A$ for every proper closed set B. By Theorem 5.5(2), there is $f \in \mathsf{c}(\bar{d})$ such that $f^{-1}(A) = A$. Let $i = 0, 1$ be such that $f(\mathcal{C}) \subseteq \mathbf{N}_{\langle i \rangle}$, and consider the map

$$j \colon \mathcal{C} \to [A]_{\mathsf{L}(\bar{d})} \colon x \mapsto A_x = (A \cap \mathbf{N}_{\langle i \rangle}) \cup \{(1-i)^\frown x\}.$$

Clearly j is an injection, so it remains only to show that $A \equiv_{\mathsf{L}(\bar{d})} A_x$ for every $x \in \mathcal{C}$. For one direction, f witnesses $A \leq_{\mathsf{L}(\bar{d})} A_x$. For the other direction, let $g_x \in \mathsf{L}(\bar{d})$ be a reduction of $\{(1-i)^\frown x\}$ to A: then $(\mathrm{id}_\mathcal{C} \upharpoonright \mathbf{N}_{\langle i \rangle}) \cup (g_x \upharpoonright \mathbf{N}_{\langle (1-i) \rangle})$ witnesses $A_x \leq_{\mathsf{L}(\bar{d})} A$.

Finally, let A be $\mathsf{L}(\bar{d})$-selfdual. Since by case assumption A is not clopen, there is an $\mathsf{L}(\bar{d})$-nonselfdual $B \neq \emptyset, \mathcal{C}$ and $n \in \omega$ such that $A \equiv_{\mathsf{L}(\bar{d})} 0^{(n)\frown}(B \oplus \neg B)$. Let $j' \colon \mathcal{C} \to [B]_{\mathsf{L}(\bar{d})}$ be an injective map: then

$$j \colon \mathcal{C} \to [A]_{\mathsf{L}(\bar{d})} \colon x \mapsto 0^{(n)\frown}(j'(x) \oplus \neg j'(x))$$

is clearly as required. $\square$

Since by Theorem 5.5 the $\mathsf{c}(\bar{d})$-hierarchy on Borel subsets of $\mathcal{C}$ is the refinement of the $\mathsf{L}(\bar{d})$-hierarchy obtained by splitting each $\mathsf{L}(\bar{d})$-selfdual degree into the singletons of its elements, using Proposition 5.6 we can represent such hierarchy as in Figure 2, where the bullets represent the $\mathsf{c}(\bar{d})$-degrees and the boxes around them represent the $\mathsf{L}(\bar{d})$-degrees they come from (notice that by Proposition 5.6(1) in the second column there are 12 different $\mathsf{c}(\bar{d})$-degrees, while in the third column we already find 240 distinct $\mathsf{c}(\bar{d})$-degrees!).

Proposition 5.6 and Theorem 5.5 also imply the following corollary.

Corollary 5.7. (1) *The $\mathsf{c}(\bar{d})$-hierarchy $\mathsf{Deg}_{\mathbf{\Delta}_1^1}(\mathsf{c}(\bar{d}))$ on Borel subsets of $\mathcal{C}$ is bad but not very bad. In fact it contains antichains of size the continuum.*
(2) *The $\mathsf{c}(\bar{d})$-hierarchy $\mathsf{Deg}_{\mathbf{\Sigma}_1^0 \cup \mathbf{\Pi}_1^0}(\mathsf{c}(\bar{d}))$ on open or closed subsets of $\mathcal{C}$ is good but not very good.*
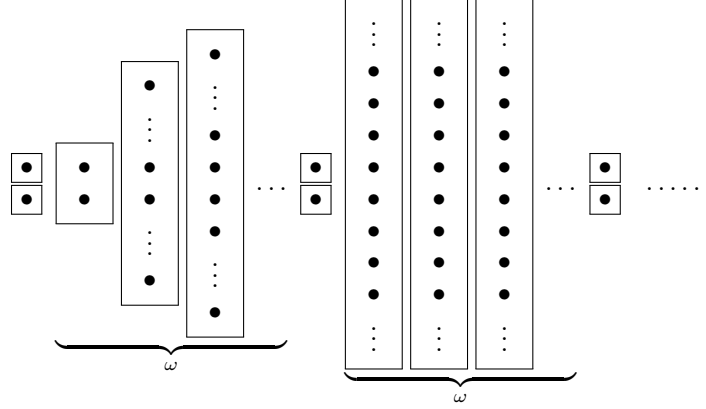
FIGURE 2. The $\mathsf{c}(\bar{d})$-hierarchy on Borel subsets of $\mathcal{C}$.

Notice that Corollary 5.7(2) gives a partial answer to [MR12, Question 6.3]. However, such solution is not completely satisfactory, as we needed to restrict our hierarchy to a very small class of subsets of $\mathcal{C}$ — of course it would be more interesting to find a reducibility $\mathcal{F}$ (on some Polish space X) inducing a good but not very good hierarchy on the entire collection of Borel subsets of X (or, under AD, even on the entire $\mathscr{P}(X)$). This last problem seems to be completely open, but the next example shows that if the requirement that the preorder inducing the hierarchy be of the form $\leq_{\mathcal{F}}$ (for some reducibility $\mathcal{F}$ on X) is dropped, then one can obtain a "natural" hierarchy on the collection of all Borel subsets of $^{\omega}\omega$ which is good but not very good .

Example 5.8. Given a set $R \subseteq \mathbb{R}^{+}$ and $A, B \subseteq {}^{\omega}\omega$ such that $A \leq_{\mathsf{Lip}(\bar{d})} B$, let $L_{A,B} = \inf\{0 < L \in \mathbb{R}^{+} \mid A \leq_{\mathsf{Lip}(\bar{d},L)} B\}$, where $\leq_{\mathsf{Lip}(\bar{d},L)}$ is as in Definition 4.13. Then set

$$A \leq_R B \iff A \leq_{\mathsf{Lip}(\bar{d})} B \wedge L_{A,B} \in R \cup \{0,1\}.$$

Notice that $\leq_R$ is always reflexive: in fact, either $A \not\leq_{\mathsf{Lip}(\bar{d},L)} A$ for all $L < 1$ (in which case the identity function witnesses $L_{A,A} = 1$), or else by considering arbitrarily large powers of any witness of $A \leq_{\mathsf{Lip}(\bar{d}),L} A$ (for some $L < 1$) we see that $L_{A,A} = 0$. In contrast, notice that in general $\leq_R$ need not to be transitive. However, when $\leq_R$ actually happens to be a preorder (as in all the relevant cases considered below), then with a little abuse of terminology we can consider the $\leq_R$-hierarchy on Borel subsets of $^{\omega}\omega$ (with the obvious definition).

Using the methods introduced at the end of [MR12, Section 4], it is easy to see that if $A, B \subseteq {}^{\omega}\omega$ are Borel sets such that $A <_{\mathsf{Lip}(\bar{d})} B$, then also $A <_R B$, because in this case $L_{A,B} = 0$. Moreover, since by [MR12, Corollary 4.4] if $A \subseteq {}^{\omega}\omega$ is $\mathsf{Lip}(\bar{d})$-nonselfdual (equivalently: $\mathsf{L}(\bar{d})$-nonselfdual), then $A \leq_{\mathsf{Lip}(\bar{d},L)} A$ for every $L > 0$, we get that for such an A, $A \leq_{\mathsf{L}(\bar{d})} B \Rightarrow A \leq_R B$ for every $B \subseteq {}^{\omega}\omega$, and if $R \subseteq (0,1]$ we in fact have that $A \leq_{\mathsf{L}(\bar{d})} B \iff A \leq_R B$. Finally, if $A \subseteq {}^{\omega}\omega$ is $\mathsf{L}(\bar{d})$-selfdual and $B \in [A]_{\mathsf{L}(\bar{d})}$, then $A \equiv_R B$ because [MR12, Proposition 4.2] implies that all the witnesses of $A \leq_{\mathsf{L}(\bar{d})} B$ and $B \leq_{\mathsf{L}(\bar{d})} A$ cannot have Lipschitz constant < 1. Summing up, we get that if $R \subseteq (0,1]$ (and $\leq_R$ is transitive), then

the $\leq_R$-hierarchy refines the $\mathsf{L}(\bar{d})$-hierarchy, and may differ from it only within the $\mathsf{Lip}(\bar{d})$-selfdual degrees.

Let us now concentrate on the canonical examples given by $R_n = (0, 2^{-n}]$ (for $n \in \omega$). It is easy to check that if $n \leq 1$, then the $\leq_R$-hierarchy coincides with the $\mathsf{L}(\bar{d})$-hierarchy. However, if $n > 1$ and A is an $\mathsf{L}(\bar{d})$-selfdual set, then

$$(\dagger) \qquad A \leq_{R_n} B \iff A \leq_{\mathsf{L}(\bar{d})} B \wedge (B \equiv_{\mathsf{L}(\bar{d})} A \vee 0^{(n)\frown} A \leq_{\mathsf{L}(\bar{d})} B).$$

Therefore the restriction of $\leq_{R_n}$ to the Borel subsets of $^\omega\omega$ is always transitive (hence a preorder), and it is also well-founded. Moreover, ($\dagger$) also implies that the antichains in $\leq_R$ have always size $\leq n$. Since e.g. $\{0^{(i+1)\frown}\omega \mid i < n\}$ is an $\leq_{R_n}$-antichain of size precisely n consisting of clopen sets, we get that for all $n \geq 3$ the $\leq_{R_n}$-hierarchy on Borel subsets of $^\omega\omega$ is good but not very good.

6. WADGE-LIKE REDUCIBILITIES UNDER THE AXIOM OF CHOICE

By (the comment following) Proposition 1.2, the $\mathsf{L}(\bar{d})$-hierarchy $\mathsf{Deg}_{\mathbf{\Delta}_1^1}(\mathsf{L}(\bar{d}))$ on the Borel subsets of $^\omega\omega$ is very good, and as already recalled the same is true for larger classes of subsets of $^\omega\omega$ if we further assume corresponding determinacy axioms. It is therefore natural to ask what happens if, instead of assuming such determinacy principles, we assume the Axiom of Choice AC or other strong choice principles.

Similar considerations apply to arbitrary Polish spaces as well. It is shown in [Sch12] that for every non-zero-dimensional Polish space X the $\mathsf{W}(X)$-hierarchy on Borel subsets of X already contains antichains of size the continuum, and in fact [IST12] shows that if e.g. $X = \mathbb{R}$ then we can also embed $(\mathscr{P}(\omega), \subseteq^*)$ into $\mathsf{Deg}_{\mathbf{\Delta}_1^1}(\mathsf{W}(X))$ (but this last result cannot be extended to arbitrary X: as explained in [MRSS12, Section 5.1], all continuous functions on the Cook continuum X are either constant or the identity, and therefore all chains of subsets of X with respect to continuous reducibility have length ≤ 2). However, [MRSS12] shows that for every Polish space X, the $\mathsf{D}_\alpha(X)$-hierarchy on Borel subsets of X (where $\mathsf{D}_\alpha(X)$ denotes the collection of all $\mathbf{\Delta}_\alpha^0$-functions from X to itself) is always very good for $\alpha \geq \omega$, and that the same is true for $\alpha \geq 3$ if X is of dimension $\neq \infty$. Also these last results extend to larger classes of subsets of X under suitable determinacy assumptions, and therefore it is meaningful to ask what happens if instead we assume AC.

Not surprisingly, it turns out that under choice all the above mentioned hierarchies of degrees (on arbitrary subsets of X) become very bad. Clearly, Borel determinacy forces us to consider non-Borel subsets of X to get such results: therefore in what follows we will concentrate only on uncountable (ultrametric) Polish spaces.

Notation 6.1. If X is a set and $A \subseteq X^2$, we denote by A_x the "vertical section" determined by $x \in X$, i.e. we set $A_x = \{y \in X \mid (x, y) \in A\}$. Moreover, for every cardinal μ we set $[X]^\mu = \{Y \subseteq X \mid |Y| = \mu\}$.

Lemma 6.2 (AC)**.** *Let μ be an infinite cardinal and X be a set of size μ. Moreover, let $\mathcal{C} \subseteq [X]^\mu$, $\mathcal{F}$ be a collection of functions from X to itself, and suppose that $|\mathcal{C}| = |\mathcal{F}| = \mu$. Then there is a set $A \subseteq X^2$ such that $A_x \cap C \nleq_\mathcal{F} A_y$ for all distinct $x, y \in X$ and all $C \in \mathcal{C}$.*

Proof. We first recursively construct a sequence $(\{A_{x,\alpha}, B_{x,\alpha} \mid x \in X\})_{\alpha < \mu}$ such that $A_{x,\alpha} \cap B_{x,\alpha} = \emptyset$, $A_{x,\alpha} \subseteq A_{x,\beta}$, $B_{x,\alpha} \subseteq B_{x,\beta}$, and $|A_{x,\alpha} \cup B_{x,\alpha}| \leq |2 \cdot \alpha|$ for all $\alpha \leq \beta < \mu$ and $x \in X$.

Fix a surjection $h \colon \mu \to \mathcal{C} \times \mathcal{F} \times X^2$, and set $A_{x,0} = B_{x,0} = \emptyset$ for all $x \in X$. Let now $0 < \alpha < \mu$, and assume that all sets of the form $A_{x,\beta}, B_{x,\beta}$ for $x \in X$ and $\beta < \alpha$ have already been defined, so that we can set $A_{x,<\alpha} = \bigcup_{\beta < \alpha} A_{x,\beta}$ and $B_{x,<\alpha} = \bigcup_{\beta < \alpha} B_{x,\beta}$. Let $(C, f, x, y) \in \mathcal{C} \times \mathcal{F} \times X^2$ be such that $h(\alpha) = (C, f, x, y)$, and let $C_0 = C \setminus (A_{x,<\alpha} \cup B_{x,<\alpha})$. Notice that $|C_0| = \mu$ because $|A_{x,<\alpha} \cup B_{x,<\alpha}| < \mu$ and $|C| = \mu$. We distinguish two cases: if $|f(C_0)| < \mu$, we choose distinct $a, b \in C_0$ such that $f(a) = f(b)$ (this is possible because $|C_0| = \mu > |f(C_0)|$), and then we set $A_{x,\alpha} = A_{x,<\alpha} \cup \{a\}$, $B_{x,\alpha} = B_{x,<\alpha} \cup \{b\}$, and $A_{z,\alpha} = A_{z,<\alpha}$, $B_{z,\alpha} = B_{z,<\alpha}$ for all $z \in X$ distinct from x. If instead $|f(C_0)| = \mu$, we pick some $a \in C_0$ with $f(a) \notin A_{y,<\alpha} \cup B_{y,<\alpha}$ (which exists because $|A_{y,<\alpha} \cup B_{y,<\alpha}| < \mu$, and hence $f(C_0) \setminus (A_{y,<\alpha} \cup B_{y,<\alpha}) \neq \emptyset$), and then we set $A_{x,\alpha} = A_{x,<\alpha} \cup \{a\}$, $B_{x,\alpha} = B_{x,<\alpha}$, $A_{y,\alpha} = A_{y,<\alpha}$, $B_{y,\alpha} = B_{y,<\alpha} \cup \{f(a)\}$, and $A_{z,\alpha} = A_{z,<\alpha}$, $B_{z,\alpha} = B_{z,<\alpha}$ for all $z \in X$ distinct from x and y. This completes the recursive step of our construction, and it is easy to check by induction on $\alpha < \mu$ that the sets $A_{x,\alpha}$, $B_{x,\alpha}$ are as required.

Finally, we set $A_x = \bigcup_{\alpha < \mu} A_{x,\alpha}$, $B_x = \bigcup_{\alpha < \mu} B_{x,\alpha}$, and $A = \{(x,y) \in X^2 \mid y \in A_x\}$, so that, in particular, $A_x \cap B_x = \emptyset$ for every $x \in X$. It is straightforward to check that the α-th step in the recursive construction above ensures that f is not a reduction of $A_x \cap C$ to A_y, because either there are $a \in A_x \cap C$ and $b \in B_x \subseteq X \setminus A_x$ such that $f(a) = f(b)$, or else there is $a \in A_x \cap C$ such that $f(a) \in B_y \subseteq X \setminus A_y$. $\square$

Theorem 6.3 (AC). *Let $X = (X, d)$ be an uncountable ultrametric Polish space. Then there is a map $\psi \colon \mathscr{P}(\omega) \to \mathscr{P}(X)$ such that for all $a, b \subseteq \omega$*

(1) *if $a \subseteq b$, then $\psi(a) \leq_{\mathsf{L}(d)} \psi(b)$;*
(2) *if $\psi(a) \leq_{\mathsf{Bor}(X)} \psi(b)$, then $a \subseteq b$.*

In particular, $(\mathscr{P}(\omega), \subseteq)$ embeds into the $\mathcal{F}$-hierarchy on X for every reducibility $\mathsf{L}(d) \subseteq \mathcal{F} \subseteq \mathsf{Bor}(X)$, hence $\mathsf{Deg}(\mathcal{F})$ is very bad.

Proof. We apply the Lemma 6.2 letting $\mu = |X| = 2^{\aleph_0}$, $\mathcal{C}$ be the set of all *uncountable* Borel subsets of X, and $\mathcal{F} = \mathsf{Bor}(X)$ be the collection of all Borel functions from X to itself. Thus we obtain a sequence of $\leq_{\mathsf{Bor}(X)}$-incomparable sets $A_n \subseteq X$ (the lemma gives more, but an ω-sequence is sufficient here). Notice that each A_n is necessarily uncountable and that $A_n \neq X$, as otherwise in both cases we would easily have $A_n \leq_{\mathsf{Bor}(X)} A_m$ for every $m \in \omega$. Now choose a sequence $(X_n)_{n \in \omega}$ of pairwise disjoint uncountable clopen balls in X, and fix a Borel isomorphism $h_n \colon X \to X_n$ for every $n \in \omega$. Given $a \subseteq \omega$, set $\psi(a) = \bigcup_{n \in a} h_n(A_n)$.

To see that ψ is as required, first suppose that $a, b \subseteq \omega$ are such that $a \subseteq b$, and for every $n \in b \setminus a$ pick a point $y_n \in X_n \setminus h_n(A_n)$ (which exists because $A_n \neq X$). Then we define $f \colon X \to X$ by setting

$$f(x) = \begin{cases} y_n & \text{if } x \in X_n \text{ for some } n \in b \setminus a, \\ x & \text{otherwise.} \end{cases}$$

Clearly f reduces $\psi(a)$ to $\psi(b)$, and it is easy to check that since d is an ultrametric and the X_n are (cl)open balls, then $f \in \mathsf{L}(d)$: therefore $\psi(a) \leq_{\mathsf{L}(d)} \psi(b)$, as required.

Now let $a, b \subseteq \omega$ be such that $\psi(a) \leq_{\mathsf{Bor}(X)} \psi(b)$, let $f \in \mathsf{Bor}(X)$ be a witness of this, and fix an arbitrary $n \in a$. Notice that $f(\psi(a)) \subseteq \psi(b) \subseteq \bigcup_{m \in b} X_m$. Since

A_n is uncountable, this means that there is $m \in b$ such that $f^{-1}(X_m) \cap X_n$ is uncountable. Fix $\bar{y} \in X \setminus A_m$: setting $C = h_n^{-1}(f^{-1}(X_m) \cap X_n)$, we get that C is an uncountable Borel set, and that the map $g \colon X \to X$ defined by

$$g(x) = \begin{cases} (h_m^{-1} \circ f \circ h_n)(x) & \text{if } x \in C, \\ \bar{y} & \text{otherwise} \end{cases}$$

witnesses $A_n \cap C \leq_{\mathsf{Bor}(X)} A_m$. By our choice of the A_n's, this implies $n = m$, whence $n \in b$. Therefore $a \subseteq b$, as required. $\qquad\square$

Remark 6.4. Notice that to get Lemma 6.2 it is enough to assume that X is a well-orderable set. Therefore, also in Theorem 6.3 we can weaken the assumption AC by just requiring that X (equivalently, any uncountable Polish space) is well-orderable.

Using essentially the same argument, one can also show that a variant of Theorem 6.3 applies to arbitrary uncountable Polish spaces X (and not only to the ultrametric ones).

Theorem 6.5 (AC). *Let X be an uncountable Polish space. Then there is a map $\psi \colon \mathscr{P}(\omega) \to \mathscr{P}(X)$ such that for every $a, b \subseteq \omega$*

    (1) if $a \subseteq b$, then $\psi(a) \leq_{\mathsf{D}_2(X)} \psi(b)$;
    (2) if $\psi(a) \leq_{\mathsf{Bor}(X)} \psi(b)$, then $a \subseteq b$.

In particular, $(\mathscr{P}(\omega), \subseteq)$ embeds into the $\mathcal{F}$-hierarchy on X for every reducibility $\mathsf{D}_2(X) \subseteq \mathcal{F} \subseteq \mathsf{Bor}(X)$, hence $\mathsf{Deg}(\mathcal{F})$ is very bad.

Proof. In the proof of Theorem 6.3, let $(X_n)_{n \in \omega}$ be a partition of X into uncountable $\mathbf{\Delta}_2^0$ sets. $\qquad\square$

Remark 6.6. In Theorem 6.5 we cannot replace $\leq_{\mathsf{D}_2(X)}$ with continuous reducibility $\leq_{\mathsf{W}(X)}$: in fact, in the Cook continuum X (which is uncountable), we cannot hope to embed $(\mathscr{P}(\omega), \subseteq)$ into $\mathsf{Deg}(\mathsf{W}(X))$ because there are no infinite chains of subsets of X (with respect to continuous reducibility).

We now aim to show that if we further assume $\mathsf{V} = \mathsf{L}$, then the map ψ of Theorems 6.3 and 6.5 can be chosen to range in the collection of $\mathbf{\Pi}_1^1$ (alternatively: $\mathbf{\Sigma}_1^1$) subsets of the given (ultrametric) Polish space: this in particular implies that the $\mathsf{L}(\bar{d})$-hierarchy on $\mathbf{\Pi}_1^1$ (respectively, $\mathbf{\Sigma}_1^1$) subsets of ${}^\omega\omega$ is very bad in L. To prove this, we will modify the recursion used in the proof of Lemma 6.2 so that membership in each of the sets can be computed in the next admissible set.

Notation 6.7. For $x, y \in {}^\omega\omega$, let $\omega_1^{x,y}$ denote the least (x,y)-admissible ordinal γ.[17] To simplify the notation, set also $\omega_1^x = \omega_1^{x,x}$.

Theorem 6.8 (Spector-Gandy). *(see [Hjo10, Theorem 5.5]) A set $A \subseteq {}^\omega\omega$ is Π_1^1 in a parameter $p \in {}^\omega\omega$ if and only if there is a Σ_1-formula $\varphi(x)$ such that*

$$x \in A \Leftrightarrow L_{\omega_1^{x,p}}[x,p] \vDash \varphi(x,p)$$

for all $x \in {}^\omega\omega$.

Lemma 6.9. *Let X be a Polish space. Then there are sets $U \subseteq {}^\omega\omega \times X$ and $V \subseteq {}^\omega\omega \times X^2$ such that:*

    (1) The projections $\mathrm{p}(U)$ and $\mathrm{p}(V)$ on the first coordinate are $\mathbf{\Pi}_1^1$ sets.

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[17]That is, $\omega_1^{x,y}$ is the least $\gamma > \omega$ such that $L_\gamma[x,y]$ is a model of Kripke-Platek set theory.

(2) U is both $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_1$ on $\mathrm{p}(U) \times X$ and V is both $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_1$ on $\mathrm{p}(V) \times X^2$.

(3) A set $A \subseteq X$ is an uncountable Borel set if and only if $A = U_x = \{y \in {}^\omega\omega \mid (x,y) \in U\}$ for some $x \in \mathrm{p}(U)$.

(4) A set $A \subseteq X^2$ is the graph of a Borel function from X to itself if and only if $A = V_x = \{(y,z) \in {}^\omega\omega \mid (x,y,z) \in V\}$ for some $x \in \mathrm{p}(V)$.

Sketch of proof. Since X is second-countable, an arbitrary Borel set $A \subseteq X$ can be easily coded by a countable well-founded tree which describes how the Borel set is built up from basic open sets using countable unions and complements. This tree and a tree coding a perfect subset of A are then coded into an element of $x \in {}^\omega\omega$, and then for such x we set $U_x = A$.

For what concerns Borel functions, notice that we can concentrate only on ultra-metric Polish spaces $X = (X, d)$, because the result can then be transferred to an arbitrary Polish space Y by using a Borel isomorphism between X and Y. Therefore from now on we fix an ultrametric Polish space $X = (X, d)$. Recall that $\mathsf{Bor}(X)$ coincide with the collection of all Baire class α functions (for arbitrary $\alpha < \omega_1$), i.e. with the closure under pointwise limits of the collection of all Lipschitz functions (see [MR09b, Corollary 2.16] and [Kec95, Theorems 24.3]). Starting with a function f defined on a fixed countable dense set $D \subseteq X$, we form the (pseudo-)limit $\bar{f}$ of f by setting $\bar{f}(x) = \lim_{n \to \infty} f(x_n)$ (for an arbitrary sequence $(x_n)_{n \in \omega}$ in D with $\lim_{n \to \infty} x_n = x$) if

$$\mathrm{osc}_f(x) = \lim_{n \to \infty} \sup\{d(f(y), f(z)) \mid y, z \in X \wedge d(x,y), d(x,z) < 2^{-n}\} = 0,$$

and $\bar{f}(x) = y_0$ (for $y_0 \in X$ a fixed value) otherwise. From a countable family of functions f as above attached to the terminal nodes of a given well-founded tree, we can then build up a Borel function g by forming (pseudo-)limits (i.e. taking the pointwise limit where it exists and some fixed value $y_0 \in X$ elsewhere) in the obvious way along the tree. The tree is then coded into an element of $x \in {}^\omega\omega$, and for all x's built in this way we let V_x be the graph of the corresponding Borel function g.

Notice that the sets of codes (for both Borel sets and Borel functions) have to be $\mathbf{\Pi}^1_1$ to express that the trees used in the coding are well-founded. $\square$

Lemma 6.10. *Assume* $\mathsf{V} = \mathsf{L}$ *and let* $X = (X, d)$ *be an uncountable Polish space. Then there is a* $\mathbf{\Pi}^1_1$ *set* $A \subseteq X^2$ *such that* $A_x \cap C \not\leq_{\mathsf{Bor}(X)} A_y$ *for all distinct* $x, y \in X$ *and all uncountable Borel sets* $C \subseteq X$.

Proof. Since every uncountable Polish space is Borel isomorphic to $({}^\omega\omega, \bar{d})$, we may assume without loss of generality that $X = (X, d) = ({}^\omega\omega, \bar{d})$. Note that the sets U and V obtained in Lemma 6.9 coding the uncountable Borel subsets of $X = {}^\omega\omega$ and the Borel functions from $X = {}^\omega\omega$ to itself are actually both Π^1_1 (without parameters).

Recall that $|X| = \omega_1$ since $\mathsf{V} = \mathsf{L}$. Let us construct a sequence $(\gamma_\alpha, X_\alpha, \{A_{x,\alpha}, B_{x,\alpha} \mid x \in X_\alpha\})_{\alpha < \omega_1}$, by recursion on $\alpha < \omega_1$, where the $A_{x,\alpha}$'s and the $B_{x,\alpha}$'s are as in the proof of Lemma 6.2 (setting $\mu = \omega_1$), γ_α is a countable admissible ordinal, and X_α is a countable (or finite) subset of X. We first let γ_0 be the least admissible ordinal and $X_0 = \emptyset$ (so that none of the $A_{x,0}, B_{x,0}$ need to be defined and $\{A_{x,0}, B_{x,0} \mid x \in X_0\} = \emptyset$). In the α-th step of the recursion (for $0 < \alpha < \omega_1$), let $s_\alpha = (\gamma_\beta, X_\beta, \{A_{x,\beta}, B_{x,\beta} \mid x \in X_\beta\})_{\beta < \alpha}$, and set $X_{<\alpha} = \bigcup_{\beta < \alpha} X_\beta$, $A_{z,<\alpha} = \bigcup\{A_{z,\beta} \mid \beta < \alpha \wedge z \in X_\beta\}$, and $B_{z,<\alpha} = \bigcup\{B_{z,\beta} \mid \beta < \alpha \wedge z \in X_\beta\}$ for all $z \in X_{<\alpha}$.

Let γ_α be the least admissible with $s_\alpha \in L_{\gamma_\alpha}$, and let c denote the α-th element in $<_L$. Check if c is of the form $c = (u, v, x, y) \in ({}^\omega\omega)^2 \times X^2$ and if the Σ_1-formula which states that u and v are codes (i.e. $u \in \mathrm{p}(U)$ and $v \in \mathrm{p}(V)$) via Theorem 6.8 holds in L_{γ_α}. If not, $A_{z,\alpha} = A_{z,<\alpha}$, and $B_{z,\alpha} = B_{z,<\alpha}$ for all $z \in X_\alpha = X_{<\alpha}$.

Now suppose instead that c is of the form above, and that u, v code C, f, i.e. that $u \in \mathrm{p}(U)$, $v \in \mathrm{p}(V)$, $C = U_u$, and V_v is the graph of f. Let $C_0 = C \setminus (A_{x,<\alpha} \cup B_{x,<\alpha})$, so that, in particular, C_0 is uncountable.

We now consider three (not necessarily mutually exclusive) cases:

(1) Let a_0 be the $<_L$-least $a \in C_0 \setminus L_{\gamma_\alpha}$ with $f(a) \in A_{y,<\alpha}$, and a_1 be the $<_L$-least $a \in C_0 \setminus L_{\gamma_\alpha}$ such that $a_0 <_L a \wedge f(a) \in A_{y,<\alpha}$, if they exist. There are such a_0 and a_1 if $f(z) \in A_{y,<\alpha}$ for uncountably many $z \in C_0$.
(2) Let a_2 be the $<_L$-least $a \in C_0 \setminus L_{\gamma_\alpha}$ with $f(a) \in B_{y,<\alpha}$, if it exist. There is such an a_2 if $f(z) \in B_{y,<\alpha}$ for uncountably many $z \in C_0$.
(3) Let a_3 be the $<_L$-least $a \in C_0 \setminus L_{\gamma_\alpha}$ with $f(a) \notin A_{y,\alpha} \cup B_{y,<\alpha}$, if it exists. There is such an a_3 if $f(z) \notin A_{y,\alpha} \cup B_{y,<\alpha}$ for uncountably many $z \in C_0$.

Notice that at least one of a_1, a_2, a_3 is defined since C_0 is uncountable, so let a denote the $<_L$-least element of $\{a_1, a_2, a_3\}$. Let us mention that to define a, it is not necessary to know which of a_1, a_2, a_3 are defined, but it is sufficient to search for the $<_L$-least a satisfying one of the conditions. We then set $X_\alpha = X_{<\alpha} \cup \{x, y\}$ and $A_{x,\alpha} = A_{x,<\alpha} \cup \{a\}$. If $a = a_1$, we further set $B_{x,\alpha} = B_{x,<\alpha} \cup \{a_0\}$, while if $a = a_3$ we further set $B_{y,\alpha} = B_{y,<\alpha} \cup \{f(a)\}$. Finally, for $z \in X_\alpha$ we set $A_{z,\alpha} = A_{z,<\alpha}$ and $B_{z,\alpha} = B_{z,<\alpha}$ in every case in which these sets have not already been explicitly defined before. This concludes the construction of the desired sequence.

Let now $A_x = \bigcup_{\alpha<\omega_1} A_{x,\alpha}$ and $A = \{(x, y) \mid y \in A_x\}$. Arguing as in the proof of Lemma 6.2, it is easy to check that the α-th step of the construction guarantees that f does not reduce $A_x \cap C$ to A_y.

To see that A is a Π_1^1 set, consider some $a \in A_x$ and find the least $\alpha < \omega_1$ with $a \in A_{x,\alpha}$. Recall that we are assuming that the elements of X are sequences of natural numbers, so it makes sense to consider the countable ordinal ω_1^a. Notice that $\gamma_\alpha < \omega_1^a$ because $a \notin L_{\gamma_\alpha}$. Recall that we picked γ_α so that $s_\alpha = (\gamma_\beta, X_\beta, \{A_{x,\beta}, B_{x,\beta} \mid x \in X_\beta\})_{\beta<\alpha} \in L_{\gamma_\alpha}$. It is then easy to check that the statement "s_α is the sequence constructed using the recursive definition above for $\beta < \alpha$" is Σ_1 over L_{γ_α}. Suppose now that $c = (u, v, x, y)$ is the α-th element in $<_L$. Note that $\alpha < \gamma_\alpha$ since $s_\alpha \in L_{\gamma_\alpha}$, and hence $c \in L_{\gamma_\alpha}$. The statement that $u \in \mathrm{p}(U)$ and $v \in \mathrm{p}(V)$ is Σ_1 over L_{γ_α} as well, while the choice of a is Δ_1 over $L_{\omega_1^a}$. Hence membership of a into A_x is equivalent to the Σ_1-over-$L_{\omega_1^a}$ statement that there is a sequence s_α constructed according to the recursion and an ordinal γ_α such that a is chosen at the α-th step of the construction. Therefore A is a Π_1^1 set by Theorem 6.8. $\qquad\square$

Arguing as in Theorem 6.3 and Corollary 6.5, we now obtain:

Theorem 6.11. *Assume* $\mathsf{V} = \mathsf{L}$ *and let* $X = (X, d)$ *be an uncountable ultrametric Polish space. Then there is a map* ψ *from* $\mathscr{P}(\omega)$ *into the* $\mathbf{\Pi}_1^1$ *subsets of* X *such that for all* $a, b \subseteq \omega$

    (1) if $a \subseteq b$, *then* $\psi(a) \leq_{\mathsf{L}(d)} \psi(b)$;
    (2) if $\psi(a) \leq_{\mathsf{Bor}(X)} \psi(b)$, *then* $a \subseteq b$.

In particular, $(\mathscr{P}(\omega), \subseteq)$ *embeds into the* $\mathcal{F}$-*hierarchy on the* $\mathbf{\Pi}_1^1$ *subsets of* X *for every reducibility* $\mathsf{L}(d) \subseteq \mathcal{F} \subseteq \mathsf{Bor}(X)$, *hence* $\mathsf{Deg}_{\mathbf{\Pi}_1^1}(\mathcal{F})$ *is very bad.*

Theorem 6.12. *Assume* $\mathsf{V} = \mathsf{L}$ *and let* X *be an uncountable Polish space. Then there is a map* ψ *from* $\mathscr{P}(\omega)$ *into the* $\mathbf{\Pi}_1^1$ *subsets of* X *such that for every* $a, b \subseteq \omega$

    (1) if $a \subseteq b$, *then* $\psi(a) \leq_{\mathsf{D}_2(X)} \psi(b)$;

    (2) if $\psi(a) \leq_{\mathsf{Bor}(X)} \psi(b)$, *then* $a \subseteq b$.

In particular, $(\mathscr{P}(\omega), \subseteq)$ *embeds into the* $\mathcal{F}$*-hierarchy on the* $\mathbf{\Pi}_1^1$ *subsets of* X *for every reducibility* $\mathsf{D}_2(X) \subseteq \mathcal{F} \subseteq \mathsf{Bor}(X)$, *hence* $\mathsf{Deg}_{\mathbf{\Pi}_1^1}(\mathcal{F})$ *is very bad.*

The existence of maps $\widetilde{\psi} \colon \mathscr{P}(\omega) \to \mathbf{\Sigma}_1^1(X)$ with the properties stated in Theorems 6.11 and 6.12 follows immediately by taking complements, i.e. by setting $\widetilde{\psi}(a) = X \setminus \psi(a)$ for every $a \subseteq \omega$ (where $\psi \colon \mathscr{P}(\omega) \to \mathbf{\Pi}_1^1(X)$ is as in Theorem 6.11 or Theorem 6.12).

Remark 6.13. By Borel determinacy, the requirement that ψ ranges into $\mathbf{\Pi}_1^1$ (alternatively: $\mathbf{\Sigma}_1^1$) subsets of X in Theorems 6.11 and 6.12 cannot be further improved, and therefore such results are optimal.

It is well-known that $\mathbf{\Pi}_1^1$-determinacy implies that e.g. the $\mathsf{L}(\bar{d})$-hierarchy on $\mathbf{\Pi}_1^1$ subsets of $^\omega\omega$ is very good. In fact, Harrington [Har78] (essentially) showed that the following are equivalent:

- every $\mathbf{\Pi}_1^1$ subset of $^\omega\omega$ is determined;
- for all $a \in {}^\omega\omega$, $a^\#$ exists;
- $\mathsf{SLO}^{\mathsf{L}(\bar{d})}$ holds for $\mathbf{\Pi}_1^1$ subsets of $^\omega\omega$.

Since sharps do not exist if $\mathsf{V} = \mathsf{L}$, Theorem 6.11 can then be regarded as a strengthening of (one direction) of the above mentioned Harrington's result: under the further assumption $\mathsf{V} = \mathsf{L}$, not only $\mathsf{SLO}^{\mathsf{L}(\bar{d})}$ for $\mathbf{\Pi}_1^1$ subsets of $^\omega\omega$ does not hold, but in fact we can embed a reasonably complicated partial order in $\mathsf{Deg}_{\mathbf{\Pi}_1^1}(\mathsf{L}(\bar{d}))$. Notice also that since $\mathsf{Deg}_{\mathbf{\Delta}_1^1}(\mathsf{L}(\bar{d}))$ needs to be very good by Borel determinacy, Theorem 6.11 actually shows that if $\mathsf{V} = \mathsf{L}$, then $(\mathscr{P}(\omega), \subseteq)$ embeds into the $\mathsf{L}(\bar{d})$-hierarchy on *proper* $\mathbf{\Pi}_1^1$ subsets of $^\omega\omega$, and Theorem 6.12 shows that the same partial order embeds also e.g. in the $\mathsf{Bor}(X)$-hierarchy on *proper* $\mathbf{\Pi}_1^1$ (alternatively: *proper* $\mathbf{\Sigma}_1^1$) subsets of any uncountable Polish space X. This conclusion considerably strengthen the well-known fact that if $\mathbf{\Pi}_1^1$-determinacy fails then there are proper $\mathbf{\Pi}_1^1$ subsets which are not (Borel-)complete for that class.

The next questions essentially asks if it is possible to further strengthen Theorems 6.3 and 6.11 by either trying to embed a more complicated quasi-order into the relevant hierarchies, or by weakening the assumption required for those results to $\mathbf{\Pi}_1^1$-determinacy.

Question 6.14. *Assume* AC.

    (1) Is there a map $\psi \colon \mathscr{P}(\omega) \to \mathscr{P}(^\omega\omega)$ *such that* $a \subseteq^* b \iff \psi(a) \leq_{\mathsf{Bor}(^\omega\omega)} \psi(b)$ *for all* $a, b \subseteq \omega$?

    (2) Does the non-existence of $0^\#$ *already imply that the* $\mathsf{Bor}(^\omega\omega)$*-hierarchy on* $\mathbf{\Pi}_1^1$ *subsets of* $^\omega\omega$ *is ill-founded?*

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